From KP/UC hierarchies to Painlevé equations

Teruhisa Tsuda

Faculty of Mathematics, Kyushu University,

Fukuoka 819-0395, Japan.

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Abstract

We study the underlying relationship between Painlevé equations and infinite-dimensional integrable systems, such as the KP and UC hierarchies. We show that a certain reduction of these hierarchies by requiring homogeneity and periodicity yields Painlevé equations, including their higher order generalization. This result allows us to clearly understand various aspects of the equations, e.g., Lax formalism, Hirota bilinear relations for τ-functions, Weyl group symmetry, and algebraic solutions in terms of the character polynomials, i.e., the Schur function and the universal character.

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1 Introduction

The present article is aimed to develop the study of Painlevé equations by means of a viewpoint of infinite-dimensional integrable systems. First of all, to explain our motivation, we recall the special polynomials associated with Painlevé equations. For example, the second one (P_{II}):

$$\frac{\mathrm{d}^2 q}{\mathrm{d}x^2} = 2q^3 + xq + a$$

has a particular solution $q \equiv 0$ when $a = 0$. Furthermore P_{II} has the Bäcklund transformations generated by (see, e.g., [13])

$$\pi : q \mapsto -q, \quad a \mapsto -a,$$

$$r_1 : q \mapsto q + \frac{a - \frac{1}{2}}{q^2 - \frac{\mathrm{d}q}{\mathrm{d}x} + \frac{x}{2}}, \quad a \mapsto 1 - a.$$

It follows that P_{II} has a rational solution if a is an integer. Interestingly enough, the factors appearing in the denominator and numerator of the rational solution form monic polynomials with integer coefficients; we call them the *Yablonskii–Vorob'ev polynomials* [54]. The first few are $T_1(x) = x$, $T_2(x) = x^3 + 4$, $T_3(x) = x^6 + 20x^3 - 80$, $T_4(x) = x(x^9 + 60x^3 + 11200)$, etc. An effective way to understand the nature of these polynomials is provided by the connection with soliton theory. Let us consider the (modified) KdV equation

$$4\frac{\partial v}{\partial t} = -6v^2 \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3}, \tag{1.1}$$

which is a typical soliton equation. Since (1.1) is a homogeneous equation (of degree −4) by counting the degree of variables as $\deg x = 1$, $\deg t = 3$, and $\deg v = -1$, it admits a similarity solution of the form $v(x, t) = (-3t/4)^{-1/3} q((-3t/4)^{-1/3} x)$. The function $q = q(x)$ thus satisfies P_{II}; see [1]. On the other hand, we have previously known that (1.1) has a rational and similarity solution written by the use of the Schur functions attached to staircase partitions. Finally, the Yablonskii–Vorob'ev polynomial turns out to be a specialization of the Schur function; see [18]. The emergence of the Schur function is significant from our standpoint because as shown by Sato [41] the KP hierarchy, which is the most basic class of soliton equations, is exactly an infinite-dimensional integrable system characterized by the Schur function.

Besides the second one, such special polynomials associated with algebraic or rational solutions have been defined also for other Painlevé equations; they are referred to as the *Okamoto polynomial* for P_{IV} and the *Umemura polynomials* collectively for P_{III}, P_{V}, and P_{VI}. These polynomials are known to possess interesting features from both combinatorial and representation-theoretical point of view (see [29]), and thus may be regarded as 'nonlinear analogues' of the classical orthogonal polynomials typified by the Jacobi polynomial. Along the same lines, the Okamoto polynomial is expressible in terms of the Schur function attached to a three-core partition, based on the fact that P_{IV} coincides with a certain similarity constraint of the Boussinesq equation (which belongs to the category of KP hierarchy as well as KdV); see [19, 31]. But, however, it takes on a different posture concerning the Umemura polynomials for P_{V} and P_{VI}. As was discovered by Masuda et al. [25, 26], they are in fact expressed by the *universal character* (attached to a pair of staircase partitions).

The universal character $S_{[\lambda,\mu]}$, defined by Koike [21], is a polynomial attached to a pair of partitions $[\lambda, \mu]$ and is a generalization of the Schur function S_λ. While the latter, as is well known, describes the character of an irreducible polynomial representation of the general linear group, the former does that of a rational one. Inspired by the connection between the KP hierarchy and the Schur function, the author proposed in [46] an extension of the KP hierarchy, called the *UC hierarchy*, as an infinite-dimensional integrable system characterized by the universal character.

Character polynomials	versus	Infinite integrable systems
Schur function S_λ		KP hierarchy
$\cap$		$\cap$
Universal character $S_{[\lambda,\mu]}$		UC hierarchy

In this paper, expanding our subject to the UC hierarchy beyond the KP hierarchy, we present a unified derivation of Painlevé equations (including their higher order analogues) from infinite-dimensional integrable systems via a certain similarity reduction. As a corollary we clarify the origin not only of the special polynomials but also of various aspects of Painlevé equations, e.g., bilinear relations for τ-functions, Weyl group symmetry, and Lax formalism.

The KP hierarchy was originally introduced as a series of nonlinear partial differential equations associated with an auxiliary linear problem. In this paper, however, we adopt an equivalent definition of the KP hierarchy due to Date–Jimbo–Kashiwara–Miwa (see [27]) by a single functional equation (called the *bilinear identity*) for an unknown function $\tau = \tau(\boldsymbol{x})$ in infinitely many time variables $\boldsymbol{x} = (x_1, x_2, \ldots)$. Likewise, the UC hierarchy can be defined by a system of two equations for an unknown function $\tau = \tau(\boldsymbol{x})$ in $(\boldsymbol{x}, \boldsymbol{y}) = (x_1, x_2, \ldots, y_1, y_2, \ldots)$. It should be noted that if we count the degree of each variable as $\deg x_n = n$ and $\deg y_n = -n$, then both hierarchies come out to be homogeneous, and thereby admit similarity solutions. In other words, the UC

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hierarchy well generalizes the KP hierarchy by taking the negative time evolutions into account besides the positive ones while keeping its homogeneity. Remarkably, the homogeneous polynomial solutions of the KP (resp. UC) hierarchy are filled with the Schur functions (resp. universal characters). We summarize below some fundamental data to illustrate difference between the KP and UC hierarchies. For details to Sects. 2 and 6.

Table 1. Comparison of KP and UC hierarchies

	KP hierarchy	UC hierarchy
Time variables	$\boldsymbol{x} = (x_1, x_2, \ldots)$ $\deg x_n = n$	$(\boldsymbol{x}, \boldsymbol{y}) = (x_1, x_2, \ldots, y_1, y_2, \ldots)$ $\deg x_n = n, \deg y_n = -n$
Dependent variables	$\tau = \tau(\boldsymbol{x})$	$\tau = \tau(\boldsymbol{x}, \boldsymbol{y})$
Bilinear identities	$\sum_{i+j=-1} X_i^- \tau \otimes X_j^+ \tau = 0$	$\sum_{i+j=-1} X_i^- \tau \otimes X_j^+ \tau$ $= \sum_{i+j=-1} Y_i^- \tau \otimes Y_j^+ \tau = 0$
Vertex operators	$\sum_{n \in \mathbb{Z}} X_n^\pm z^n = e^{\pm\xi(\boldsymbol{x},z)} e^{\mp\xi(\tilde{\partial}_x, z^{-1})}$	$\sum_{n \in \mathbb{Z}} X_n^\pm z^n = e^{\pm\xi(\boldsymbol{x}-\partial_y,z)} e^{\mp\xi(\tilde{\partial}_x, z^{-1})}$, $\sum_{n \in \mathbb{Z}} Y_n^\pm z^{-n} = e^{\pm\xi(\boldsymbol{y}-\tilde{\partial}_x, z^{-1})} e^{\mp\xi(\partial_y, z)}$
Homogeneous polynomial solutions	$S_\lambda = S_\lambda(\boldsymbol{x})$	$S_{[\lambda,\mu]} = S_{[\lambda,\mu]}(\boldsymbol{x}, \boldsymbol{y})$
Phase space	SGM: Sato Grassmannian	$SGM \times SGM$

Moreover, we can derive from the original ones similar bilinear identities among solutions generated by successive application of vertex operators; let $\tau_n = \tau_n(\boldsymbol{x})$ and $\tau_{m,n} = \tau_{m,n}(\boldsymbol{x}, \boldsymbol{y})$ denote such sequences of solutions for the KP and UC hierarchies, respectively. In particular, the set of functional equations satisfied by the contiguous solutions is called the *modified* hierarchy. For instance, the modified KP hierarchy consists of the equation

$$\sum_{i+j=-2} X_i^- \tau_n \otimes X_j^+ \tau_{n+1} = 0.$$

Also, a typical equation of the modified UC hierarchy is (cf. Example 6.3)

$$\tau_{m,n} \otimes \tau_{m+1,n+1} = \sum_{i+j=0} X_i^- \tau_{m+1,n} \otimes X_j^+ \tau_{m,n+1}.$$

The modified hierarchies play an essential role in the relationship to Painlevé equations.

Principle ingredients of the similarity reduction are *homogeneity*, *periodicity*, and *specialization*. Firstly, since the (modified) KP and UC hierarchies are homogeneous, it is possible to restrict them to the self-similar solutions, i.e., ones that satisfy

$$E\tau_n = d_n \tau_n \quad \text{and} \quad E'\tau_{m,n} = d_{m,n} \tau_{m,n} \tag{1.2}$$

with some constants $d_n, d_{m,n} \in \mathbb{C}$, where

$$E = \sum_{n=1}^\infty n x_n \frac{\partial}{\partial x_n} \quad \text{and} \quad E' = \sum_{n=1}^\infty \left(n x_n \frac{\partial}{\partial x_n} - n y_n \frac{\partial}{\partial y_n} \right)$$

for the KP and UC cases, respectively. Secondly we reduce the dimension of phase space by imposing the periodic condition on the dependent variables as $\tau_{n+\ell} = \tau_n$ and $\tau_{m+\ell_1, n+\ell_2} = \tau_{m,n}$. Finally, in order to obtain ordinary differential equations, we need to choose a suitable direction of time flow, i.e., specialization of independent variables. The result is stated as follows:

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Theorem 1.1. *The* (*higher order*) *Painlevé equations* P_{II}, $P(A_{\ell-1}^{(1)})$, *and* P_{III}-*chain of order* $2\ell - 2$ *can be obtained as a certain similarity reduction of the modified KP hierarchy with period of order* 2, ℓ (≥ 3), *and* ℓ (≥ 2), *respectively. Likewise, both* $P(A_{2\ell-1}^{(1)})$ *and* P_{VI}-*chain of order* $2\ell - 2$ ($\ell \geq 2$) *can be obtained as that of the modified UC hierarchy with* (ℓ, ℓ)-*periodicity.*

(See Tables 2 and 3 below.)

Here the symbol $P(A_{\ell-1}^{(1)})$ represents the higher order Painlevé equation of type $A_{\ell-1}^{(1)}$ ([30]) or, equivalently, the Darboux chain with period ℓ ([3, 53]); this is a further generalization of P_{IV} and P_{V}, and indeed recovers the original ones if $\ell = 3$ and $\ell = 4$, respectively. The P_{III}-chain is a higher order analogue of P_{III} ([4, 44, 55]). According to its Lax pair (see Sect. 8.3), the P_{VI}-chain is identified as a certain subfamily of the Schlesinger systems ([43]). Both P_{III}- and P_{VI}-chains literally include the originals as their lowest order members. Note that part of Theorem 1.1 about the reduction of the KP hierarchy to P_{II} and $P(A_{\ell-1}^{(1)})$ has been known from [1] and [28, 42]; see also Remark 1.2. The present result generalizes these previous ones to cover all the classical six Painlevé equations (including their higher order analogues) by involving the UC hierarchy besides the KP hierarchy. We emphasize that our study is based *only* on the bilinear identities of the KP and UC hierarchies, which play the central role as master equations in soliton theory.

Now, let us describe some advantages of the similarity reduction in Painlevé equations.

(i) (Bilinear form of Painlevé equations). In contrast with its original nonlinear one, an alternative quadratic expression of a Painlevé equation is often called the (*Hirota*) *bilinear form*. Via the similarity reduction, the bilinear forms can be reduced directly from the bilinear identities of the KP and UC hierarchies, which are *a priori* quadratic relations.

(ii) (Algebraic solutions in terms of the character polynomials). Since the Schur function and universal character are the homogeneous polynomial solutions of the KP and UC hierarchies, respectively, they are consistent with the similarity reduction and thus give rise to algebraic solutions of Painlevé equations.

(iii) (Weyl group symmetry). A sequence of homogeneous solutions of the KP hierarchy admits an action of a Weyl group of type A generated by a permutation of two serial vertex operators at each site. Likewise, for the UC case we have a commutative pair of Weyl group actions of type A; this distinction, by the way, reflects the presence of two kinds of vertex operators $\{X_i^{\pm}\}$ and $\{Y_i^{\pm}\}$, which commute with each other. The above explains one origin of Weyl group symmetry of Painlevé equations.

(iv) (Lax formalism). The KP hierarchy amounts to the complete integrability condition of a system of linear equations, whose dependent variables are

$$\psi_n(\boldsymbol{x}, k) = \frac{\tau_n(\boldsymbol{x} - [k^{-1}])}{\tau_n(\boldsymbol{x})} e^{\xi(\boldsymbol{x}, k)},$$

where $[k] = (k, k^2/2, k^3/3, \ldots)$ and $\xi(\boldsymbol{x}, k) = \sum_{n=1}^{\infty} x_n k^n$. We call $\psi_n = \psi_n(\boldsymbol{x}, k)$ a *wave function* and an extra parameter k a *spectral variable*; see, e.g., [27]. The wave functions are extendedly defined for the UC hierarchy also as

$$\psi_{m,n}(\boldsymbol{x}, \boldsymbol{y}, k) = \frac{\tau_{m,n-1}(\boldsymbol{x} - [k^{-1}], \boldsymbol{y} - [k])}{\tau_{m,n}(\boldsymbol{x}, \boldsymbol{y})} e^{\xi(\boldsymbol{x}, k)}.$$

The bilinear identities generate the linear equations for the wave functions. Under the similarity reduction they naturally induce an associated linear system with a Painlevé equation, i.e., a *Lax pair*; one of which is the linear ordinary differential equation with respect to the spectral variable k and the other governs its monodromy preserving deformation. Note that, by this means, compatibility of the Lax pair is *a priori* established.

Among the most importance is the Lax formalism because: it enables us not only to classify the resulting Painlevé equation by singularity type of its associated linear equation but also to detect an appropriate dependent variable to translate its bilinear form into nonlinear one.

The following tables 2 and 3 show the corresponding choice to each individual Painlevé equation, i.e., the periodicity and specialization of time variables imposed on the KP and UC hierarchies. We also indicate (in the fourth column of each table) the number of singularities, and *Poincaré rank* for an irregular one, of the associated linear ordinary differential equation with respect to the spectral variable. Concerning the specialization (see the second column), we note that s or t is converted to the independent variable of a Painlevé equation through the similarity reduction; in parallel, a and b, together with d_n and $d_{m,n}$ appearing in (1.2), go over to constant parameters of it.

Table 2. From KP hierarchy to Painlevé equations

Period	Specialization on x	Painlevé eq.	Linear ODE in $z = k^\ell$	Ref.
$\ell = 2$	$x_n = 0 \ (n \neq 1, 3)$	P_{II}	2×2-system with 1 reg. sing. 1 irreg. sing. (rk $= 3/2$)	Sect. 5; cf. [1]
$\ell \ (\geq 3)$	$x_n = 0 \ (n \neq 1, 2)$	$P(A_{\ell-1}^{(1)})$: $\ell = 3 \Rightarrow P_{\mathrm{IV}}$, $\ell = 4 \Rightarrow P_{\mathrm{V}}$	$\ell \times \ell$-system with 1 reg. sing. 1 irreg. sing. (rk $= 2/\ell$)	Sect. 4; cf. [28]
$\ell \ (\geq 2)$	$x_1 = s + a$, $x_n = a/n \ (n \neq 1)$	P_{III}-chain: $\ell = 2 \Rightarrow P_{\mathrm{III}}$	$\ell \times \ell$-system with 2 reg. sing. 1 irreg. sing. (rk $= 1/\ell$)	Sect. 3

Table 3. From UC hierarchy to Painlevé equations

Period	Specialization on (x, y)	Painlevé eq.	Linear ODE in $z = k^\ell$	Ref.
(ℓ, ℓ)	$x_n = s + a/n$, $y_n = -s + a/n$	$P(A_{2\ell-1}^{(1)})$: $\ell = 2 \Rightarrow P_{\mathrm{V}}$	$\ell \times \ell$-system with 2 reg. sing. 1 irreg. sing. (rk $= 1$)	Sect. 7
(ℓ, ℓ)	$x_n = (a + bt^n)/n$, $y_n = (a + bt^{-n})/n$	P_{VI}-chain: $\ell = 2 \Rightarrow P_{\mathrm{VI}}$	$\ell \times \ell$-system with 4 reg. sing. (Fuchsian)	Sect. 8

Remark 1.2. Since the pioneering work of Ablowitz and Segur [1], the similarity reduction from the KdV equation to P_{II} has been well known. The connection between $P(A_{\ell-1}^{(1)})$ and the KP hierarchy was first pointed out by Schiff [42], and it was studied independently by Noumi and Yamada (see, e.g., [28]) from a group-theoretical point of view. Their theory still has been developed with involving the Drinfel'd–Sokolov hierarchy ([7]) and achieved various higher order Painlevé equations; see [9, 10, 11, 20, 32, 40]. It would be an interesting and important problem to examine their relevance to our present results.

Remark 1.3. The present work mainly deals with nonlinear ordinary differential equations of isomonodromic type. However, our approach remains valid for the partial differential case. In [51], we explore the similarity reduction from the UC hierarchy to a broad class of the Schlesinger systems including the P_{VI}-chain and the Garnier system. See also [50] for their hypergeometric solutions.

In the rest of this paper, we investigate each of Painlevé equations on the basis of the KP and UC hierarchies. In the next section, we begin by a brief review about the KP hierarchy and then construct a sequence of its homogeneous solutions applying the vertex operator technique. We present a Weyl group symmetry of type A acting on this sequence and arrange some useful formulae for the following three sections. In Sect. 3, we derive the P_{III}-chain from the (modified) KP hierarchy through a similarity reduction. We show how its bilinear form, Weyl group symmetry, rational solutions expressed in terms of the Schur function, and Lax formalism are systematically created. Similarity reductions of the KP hierarchy to $P(A_{\ell-1}^{(1)})$ and P_{II} are the subjects of Sects. 4 and 5, respectively; though the result itself is essentially known, we demonstrate concisely the reduction to clarify our simple idea that various aspects of Painlevé equations originate from the bilinear identities. Section 6 provides an introduction of the UC hierarchy, which is an extension of the KP hierarchy, with preliminaries to the last two sections. Similarity reduction of the (modified) UC hierarchy to $P(A_{2\ell-1}^{(1)})$ is studied in Sect. 7. Interestingly enough, we find a Lax pair of $P(A_{2\ell-1}^{(1)})$ different in both size and singularity type from that given in Sect. 4; cf. Tables 2 and 3. In Sect. 8, we produce the P_{VI}-chain, a higher order analogue of P_{VI}, as a reduction of the UC hierarchy. The corresponding linear equation is a Fuchsian system with four regular singularities; thus, the P_{VI}-chain is equivalent to a particular case of the Schlesinger systems. In the appendix, we briefly indicate a derivation of P_I from the KP hierarchy by using a Virasoro operator.

2   KP hierarchy

In this section, we recall some basic facts about the KP hierarchy starting from the vertex operators associated with the Schur function. Two contiguous solutions connected by the vertex operators satisfy a certain bilinear relation, which we call the *modified* KP hierarchy; it will be crucial for investigating the link to Painlevé equations. We consider a sequence of homogeneous solutions of the hierarchies and present its Weyl group symmetry of type A. Some relevant formulae are also prepared for the following three sections.

2.1   Schur function, vertex operator and KP hierarchy

We begin by recalling the definition of the Schur function and then introduce the vertex operators which play roles of raising operators of it. A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a sequence of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and that $\lambda_i = 0$ for $i \gg 0$. The number $l = l(\lambda) = \{i \,|\, \lambda_i \neq 0\}$ is called the *length* of λ; and the sum $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_l$ is called the *weight* of λ. The *Schur function* $S_\lambda = S_\lambda(\boldsymbol{x})$ attached to a partition λ is a polynomial in $\boldsymbol{x} = (x_1, x_2, \ldots)$ determined by the Jacobi–Trudi formula (see, e.g., [23]):

$$S_\lambda(\boldsymbol{x}) = \det\left(p_{\lambda_i - i + j}(\boldsymbol{x})\right)_{1 \leq i, j \leq l}, \tag{2.1}$$

where p_n ($n \in \mathbb{Z}$) is defined by the generating function

$$\sum_{n \in \mathbb{Z}} p_n(\boldsymbol{x}) k^n = e^{\xi(\boldsymbol{x},k)} \quad \text{and} \quad \xi(\boldsymbol{x}, k) = \sum_{n=1}^{\infty} x_n k^n \tag{2.2}$$

or, equivalently, $p_n = 0$ ($n < 0$), $p_0 = 1$, and

$$p_n = \sum_{k_1 + 2k_2 + \cdots + nk_n = n} \frac{x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}}{k_1! k_2! \cdots k_n!}.$$

If we count the degree of variable x_n as $\deg x_n = n$, then S_λ is a (weighted) homogeneous polynomial of degree $|\lambda|$.

Introduce the partial differential operators

$$X^{\pm}(k) = \sum_{n \in \mathbb{Z}} X_n^{\pm} k^n = e^{\pm\xi(\boldsymbol{x},k)} e^{\mp\xi(\tilde{\partial}_{\boldsymbol{x}}, k^{-1})}, \tag{2.3}$$

called the *vertex operators*. Here $\tilde{\partial}_{\boldsymbol{x}}$ stands for $\left(\frac{\partial}{\partial x_1}, \frac{1}{2}\frac{\partial}{\partial x_2}, \frac{1}{3}\frac{\partial}{\partial x_3}, \ldots\right)$. It is worth mentioning that the operator X_n^+ is a raising operator of the Schur function in the following sense:

$$S_\lambda(\boldsymbol{x}) = X_{\lambda_1}^+ \ldots X_{\lambda_l}^+ .1 \tag{2.4}$$

for a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$. Let us now formulate the KP hierarchy by using $X_n^{\pm}$.

Definition 2.1. For an unknown function $\tau = \tau(\boldsymbol{x})$, the bilinear equation

$$\sum_{i+j=-1} X_i^- \tau \otimes X_j^+ \tau = 0 \tag{2.5}$$

is called the *KP hierarchy*.

We regard $f \otimes g = f(\boldsymbol{x}')g(\boldsymbol{x})$ as an element of $\mathbb{C}[\boldsymbol{x}'] \otimes \mathbb{C}[\boldsymbol{x}]$. It is then obvious that (2.5) can be rewritten into the equation

$$\frac{1}{2\pi\sqrt{-1}} \oint e^{\xi(\boldsymbol{x}-\boldsymbol{x}',z)} dz\, \tau(\boldsymbol{x}' + [z^{-1}])\tau(\boldsymbol{x} - [z^{-1}]) = 0 \tag{2.6}$$

with $\boldsymbol{x}$ and $\boldsymbol{x}'$ being arbitrary parameters. Here the symbol $[t]$ denotes $(t, t^2/2, t^3/3, \ldots)$ and the integration $\oint \frac{dz}{2\pi\sqrt{-1}}$ means taking the coefficient of $1/z$ of the integrand as a (formal) Laurent expansion in z. By choosing appropriately the specialization of the arbitrary parameters $\boldsymbol{x}$ and $\boldsymbol{x}'$ in (2.6), we can derive various functional equations for τ. For instance, the Taylor expansion at $\boldsymbol{x} = \boldsymbol{x}'$ yields an infinite series of nonlinear differential equations (cf. Sect. 2.3), the first member of which is

$$\left(D_{x_1}^{\ 4} + 3D_{x_2}^{\ 2} - 4D_{x_1} D_{x_3}\right) \tau \cdot \tau = 0. \tag{2.7}$$

Here recall the definition of the *Hirota differential*

$$P(D_{\boldsymbol{x}})f(\boldsymbol{x}) \cdot g(\boldsymbol{x}) = P(\partial_{\boldsymbol{a}})f(\boldsymbol{x} + \boldsymbol{a})g(\boldsymbol{x} - \boldsymbol{a})|_{\boldsymbol{a}=0} \tag{2.8}$$

for a polynomial $P(D_{\boldsymbol{x}})$ in $D_{\boldsymbol{x}} = (D_{x_1}, D_{x_2}, \ldots)$. Through the change of variables

$$u = 2\left(\frac{\partial}{\partial x_1}\right)^2 \log \tau, \tag{2.9}$$

in fact, (2.7) is converted to the KP (Kadomtsev–Petviashvili) equation

$$\frac{3}{4}\frac{\partial^2 u}{\partial x_2{}^2} = \frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_3} - \frac{3}{2}u\frac{\partial u}{\partial x_1} - \frac{1}{4}\frac{\partial^3 u}{\partial x_1{}^3}\right).$$

As one can read from (2.9), it is quite natural to require $\tau(\boldsymbol{x})$ to be an entire function. We call an entire function $\tau(\boldsymbol{x})$ solving the KP hierarchy a τ-*function*. Note that τ-functions are distinguished up to constant multiplication. The celebrated Sato theory revealed that each solution of the KP hierarchy is parameterized by a point of an infinite-dimensional Grassmann manifold, called the *Sato Grassmannian*, and the τ-functions emerge from its Plücker coordinates; all the Hirota differential equations, like (2.7), constituting the KP hierarchy are identified with the Plücker relations. Moreover, from the viewpoint of infinite-dimensional Lie algebra, the τ-functions can be described as the orbit of a trivial one $\tau \equiv 1$ under the action of vertex operators; see, e.g., [27].

Remark 2.2 (Scaling symmetry of the KP hierarchy). If $\tau = \tau(x_1, x_2, x_3, \ldots)$ is a solution of the KP hierarchy, (2.5), then so is $\tau(cx_1, c^2 x_2, c^3 x_3, \ldots)$ for any $c \in \mathbb{C}^\times$; thus, it is meaningful to take interest in solutions invariant under this scaling symmetry up to constant multiplication. For example, the Schur function $S_\lambda(\boldsymbol{x})$ is a solution of the KP hierarchy, and possesses the homogeneity $S_\lambda(cx_1, c^2 x_2, c^3 x_3, \ldots) = c^{|\lambda|}S_\lambda(\boldsymbol{x})$. In other words, $ES_\lambda(\boldsymbol{x}) = |\lambda|S_\lambda(\boldsymbol{x})$ with E denoting the Euler operator:

$$E = \sum_{n=1}^{\infty} nx_n\frac{\partial}{\partial x_n}. \tag{2.10}$$

It should be noted that the whole set of homogeneous polynomial solutions of the KP hierarchy is equal to that of the Schur functions; see [41].

Now let us put our attention to the functional relations for a sequence of τ-functions connected by successive application of vertex operators. Suppose $\tau_0 := \tau(\boldsymbol{x})$ to be a solution of the KP hierarchy, (2.5). Let $\tau_1 := X^+(\alpha)\tau(\boldsymbol{x})$ with an arbitrary constant $\alpha \in \mathbb{C}^\times$. Then τ_1 solves (2.5) again. Moreover we can deduce the bilinear equation $\sum_{i+j=-2} X_i^- \tau_0 \otimes X_j^+ \tau_1 = 0$ from (2.5) multiplied by $1 \otimes X^+(\alpha)$ with the aid of the *fermionic* relations

$$X_i^\pm X_j^\pm + X_{j-1}^\pm X_{i+1}^\pm = 0 \quad \text{and} \quad X_i^+ X_j^- + X_{j+1}^- X_{i-1}^+ = \delta_{i+j,0}. \tag{2.11}$$

Definition 2.3. For a sequence $\tau_n = \tau_n(\boldsymbol{x})$ $(n \in \mathbb{Z})$ of unknown functions, the system of bilinear equations

$$\sum_{i+j=-2} X_i^- \tau_n \otimes X_j^+ \tau_{n+1} = 0 \tag{2.12}$$

is called the *modified KP hierarchy* or, shortly, *mKP hierarchy*.

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2.2 A homogeneous τ-sequence and its Weyl group symmetry

Define a partial differential operator $V(c)$ $(c \in \mathbb{C})$ by

$$V(c) = \int_\gamma X^+(k)k^{-c-1}\mathrm{d}k,$$

where the path $\gamma : [0, 1] \to \mathbb{C}$ is well chosen such that $[X^+(k)k^{-c}]_{\gamma(0)}^{\gamma(1)} = 0$; thus, γ may depend on c in general. One can verify in the same way as Sect. 2.1 that: if $\tau_0(x)$ is a solution of the KP hierarchy then so is $\tau_1 = V(c)\tau_0$ and, moreover, the pair (τ_0, τ_1) satisfies (2.12). In this sense we may call also $V(c)$ a vertex operator. An interesting feature of our vertex operator $V(c)$ is its homogeneous property: if a function $f = f(x)$ is an eigenfunction of the Euler operator (see (2.10)), i.e., $Ef = df$ for some $d \in \mathbb{C}$, then a new function $g = V(c)f$ satisfies $Eg = (d + c)g$ again. This fact is an immediate consequence of the following

Lemma 2.4. *It holds that* $[E, V(c)] = cV(c)$ *for any* $c \in \mathbb{C}$.

Proof. First recall the formula $e^A B e^{-A} = e^{\mathrm{ad}(A)}B = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \cdots$ for any operators A and B, where $\mathrm{ad}(A)(B) = [A, B]$. Since $[\xi(x, k), \partial/\partial x_n] = -k^n$, we have $[\xi(x, k), E] = -\sum_{n \geq 1} nx_n k^n = -k\frac{\partial}{\partial k}\xi(x, k)$. Therefore

$$e^{\xi(x,k)}Ee^{-\xi(x,k)} = E - k\frac{\partial}{\partial k}\xi(x, k). \tag{2.13}$$

On the other hand, from $\left[\xi(-\tilde{\partial}_x, k^{-1}), x_n\right] = -k^{-n}/n$, we observe $\left[\xi(-\tilde{\partial}_x, k^{-1}), E\right] = -\sum_{n \geq 1} k^{-n}\frac{\partial}{\partial x_n} = -k\frac{\partial}{\partial k}\xi(-\tilde{\partial}_x, k^{-1})$. Accordingly, we obtain

$$e^{\xi(-\tilde{\partial}_x, k^{-1})}Ee^{-\xi(-\tilde{\partial}_x, k^{-1})} = E - k\frac{\partial}{\partial k}\xi(-\tilde{\partial}_x, k^{-1}). \tag{2.14}$$

Hence we see that

$$EX^+(k) = Ee^{\xi(x,k)}e^{\xi(-\tilde{\partial}_x, k^{-1})} \tag{2.15}$$

$$= e^{\xi(x,k)}Ee^{\xi(-\tilde{\partial}_x, k^{-1})} + \left(k\frac{\partial}{\partial k}e^{\xi(x,k)}\right)e^{\xi(-\tilde{\partial}_x, k^{-1})}, \qquad \text{using (2.13)}$$

$$= e^{\xi(x,k)}\left(e^{\xi(-\tilde{\partial}_x, k^{-1})}E + k\frac{\partial}{\partial k}e^{\xi(-\tilde{\partial}_x, k^{-1})}\right) + \left(k\frac{\partial}{\partial k}e^{\xi(x,k)}\right)e^{\xi(-\tilde{\partial}_x, k^{-1})}, \qquad \text{using (2.14)}$$

$$= X^+(k)E + k\frac{\partial}{\partial k}X^+(k). \tag{2.16}$$

Finally, we conclude that

$$[E, V(c)] = \int_\gamma [E, X^+(k)]k^{-c-1}\mathrm{d}k = \int_\gamma \frac{\partial X^+(k)}{\partial k}k^{-c}\mathrm{d}k, \qquad \text{using (2.16)}$$

$$= [X^+(k)k^{-c}]_{\gamma(0)}^{\gamma(1)} + c\int_\gamma X^+(k)k^{-c-1}\mathrm{d}k = cV(c)$$

via integration by parts. $\qquad\qquad\square$

Suppose $\tau_0(\boldsymbol{x})$ to be a solution of the KP hierarchy (2.5) satisfying $E\tau_0 = d_0\tau_0$. Introduce a sequence $\{\tau_0, \tau_1, \tau_2, \ldots\}$ of solutions defined recursively by $\tau_{n+1} = V(c_n)\tau_n$ for arbitrary parameters $c_n \in \mathbb{C}$ given:

$$\cdots \xrightarrow{V(c_{n-2})} \tau_{n-1} \xrightarrow{V(c_{n-1})} \tau_n \xrightarrow{V(c_n)} \tau_{n+1} \xrightarrow{V(c_{n+1})} \cdots \tag{2.17}$$

Since τ_0 is homogeneous, so are all τ_n ($n \geq 1$). To be specific, we have $E\tau_n = d_n\tau_n$ with $d_{n+1} = d_n + c_n$. We shall call a sequence of solutions of the KP hierarchy of the form (2.17) a *homogeneous τ-sequence*.

Example 2.5 (A sequence of Schur functions). For example, let $c = n$ be an integer and γ a positively oriented small circle around $k = 0$. Then $V(n) = 2\pi \sqrt{-1}X_n^+$ according to (2.3). Fix a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, and recall (2.4). Starting from a trivial solution $\tau = S_\emptyset(\boldsymbol{x}) \equiv 1$ of the KP hierarchy, we thus have a sequence of Schur functions connected by successive application of X_n^+ of the form (cf. [16]):

$$S_\emptyset = 1 \xrightarrow{X_{\lambda_l}^+} S_{(\lambda_l)} \xrightarrow{X_{\lambda_{l-1}}^+} S_{(\lambda_{l-1},\lambda_l)} \xrightarrow{X_{\lambda_{l-2}}^+} \cdots \xrightarrow{X_{\lambda_1}^+} S_{(\lambda_1,\lambda_2,\ldots,\lambda_l)}$$

This type of homogeneous τ-sequence in fact yields a class of rational or algebraic solutions of Painlevé equations; cf. [31]. We shall explicitly demonstrate for the P_{III}-chain in Sect. 3.4.

Let us now concern Weyl group symmetry of the homogeneous τ-sequence. First notice that the fermionic relation

$$V(a)V(b) + V(b-1)V(a+1) = 0 \tag{2.18}$$

holds. Interchange the $(n-1)$th and nth operations in the chain (2.17) while taking (2.18) into account. We hence obtain a new sequence

$$\cdots \xrightarrow{V(c_{n-2})} \tau_{n-1} \xrightarrow{V(c_n+1)} \hat{\tau}_n \xrightarrow{V(c_{n-1}-1)} \tau_{n+1} \xrightarrow{V(c_{n+1})} \cdots$$

which is identical with the original one, (2.17), except τ_n is replaced by

$$\hat{\tau}_n = V(c_n + 1)\tau_{n-1}.$$

Besides, the degree of $\hat{\tau}_n$ reads as $\hat{d}_n = d_{n-1} + c_n + 1 = d_{n-1} - d_n + d_{n+1} + 1$. Let us refer to the above permutation of vertex operators as r_n.

We can consider the operation r_i at each ith site in the chain. With respect to the variables $\alpha_i = \hat{d}_i - d_i = d_{i-1} - 2d_i + d_{i+1} + 1$, the operation r_i induces the transformation

$$r_i(\alpha_i) = -\alpha_i, \quad r_i(\alpha_{i\pm1}) = \alpha_{i\pm1} + \alpha_i, \quad \text{and} \quad r_i(\alpha_j) = \alpha_j \quad (j \neq i, i \pm 1).$$

This is exactly the canonical realization of a generator of the Weyl group of type A if we regard α_i as a simple root. One can easily verify that $\langle r_i \rangle$ indeed fulfills the fundamental relations

$$r_i^2 = 1, \quad r_i r_{i\pm1} r_i = r_{i\pm1} r_i r_{i\pm1}, \quad \text{and} \quad r_i r_j = r_j r_i \quad (j \neq i, i \pm 1).$$

In summary, we find a realization of the Weyl group of type A generated by a permutation r_i of two serial vertex operators at each ith site of the homogeneous τ-sequence (2.17). This is in fact one origin of Weyl group symmetry of Painlevé equations, as demonstrated lucidly for P_{III}-chain,

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$P(A_{\ell-1}^{(1)})$, and P_{II} respectively in Sects. 3.3, 4.2, and 5.2 below. Note that in [56] a similar treatment of the above Weyl group action was explained in the context of binary Darboux transformations. Also, there is an alternative approach to the symmetry of Painlevé equations based on the Gauß decomposition of (Lax) matrices; see [28].

Before closing this subsection, we shall prepare some useful formulae that will be employed later.

Lemma 2.6. *Let* $\{\ldots, \tau_{n-1}, \tau_n, \tau_{n+1}, \ldots\}$ *be a (homogeneous)* τ-*sequence such that* $\tau_{n+1} = V(c_n)\tau_n$. *Let* $\hat{\tau}_n = V(c_n + 1)\tau_{n-1}$. *Then we have*

$$\tau_{n-1} \otimes \tau_{n+1} = \sum_{i+j=-1} X_i^- \hat{\tau}_n \otimes X_j^+ \tau_n, \tag{2.19a}$$

$$\tau_n \otimes \hat{\tau}_n - \hat{\tau}_n \otimes \tau_n = \sum_{i+j=1} X_i^- \tau_{n+1} \otimes X_j^+ \tau_{n-1}. \tag{2.19b}$$

Proof. By applying $V(c_n + 2) \otimes 1$ to (2.12) one can verify (2.19a) straightforwardly via (2.11). Similarly it follows from (2.5) applied by $V(c_{n-1}) \otimes 1$ that

$$\tau_{n-1} \otimes \tau_n = \sum_{i+j=0} X_i^- \tau_n \otimes X_j^+ \tau_{n-1}. \tag{2.20}$$

In addition, applying $V(c_n + 1) \otimes 1$ to this leads to (2.19b). □

2.3 Preliminaries for Sects. 3–5: difference/differential equations inside mKP hierarchy

The mKP hierarchy (2.12) can be equivalently rewritten into

$$\frac{1}{2\pi\sqrt{-1}} \oint z e^{\xi(x-x',z)} \mathrm{d}z\, \tau_n(x' + [z^{-1}])\tau_{n+1}(x - [z^{-1}]) = 0 \tag{2.21}$$

for arbitrary x and x'. We first present some difference and/or differential equations arising from (2.21); cf. [6]. We henceforth require a solution $\tau_n(x)$ of (2.21) to be an entire function.

Lemma 2.7. *The following functional equations hold*:

$$(tD_{x_1} + 1)\, \tau_n\, (x - [t]) \cdot \tau_{n+1}\, (x) - \tau_n\, (x)\, \tau_{n+1}\, (x - [t]) = 0, \tag{2.22a}$$

$$(D_{\delta_t} - 1)\, \tau_n(x) \cdot \tau_{n+1}(x) + \tau_n\, (x - [t])\, \tau_{n+1}\, (x + [t]) = 0, \tag{2.22b}$$

$$(t - s)\tau_n(x - [t] - [s])\tau_{n+1}(x) - t\tau_n(x - [t])\tau_{n+1}(x - [s]) + s\tau_n(x - [s])\tau_{n+1}(x - [t]) = 0, \tag{2.22c}$$

$$\left(D_{\delta_t} + \frac{t}{s-t}\right)\tau_n(x - [s]) \cdot \tau_{n+1}(x) + \frac{t}{t-s}\tau_n(x - [t])\tau_{n+1}(x + [t] - [s]) = 0, \tag{2.22d}$$

where D_{δ_t} *denotes the Hirota differential with respect to the vector field*

$$\delta_t = \sum_{n=1}^{\infty} t^n \frac{\partial}{\partial x_n}. \tag{2.23}$$

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Proof. We shall verify (2.22a), (2.22b), (2.22c), and (2.22d) by taking the variables in (2.21) as

(a) $x - x' = [t]$,   (b) $x - x' = 2[t]$,   (c) $x - x' = [t] + [s]$,   (d) $x - x' = 2[t] + [s]$,

respectively. Let $F = F(z)$ denote the integrand of (2.21) and write $\Omega = z e^{\xi(x-x',z)} dz$, for convenience.

Substitute $x' = x - [t]$ for the case (a). Under this specialization we observe

$$\Omega = \frac{z \, dz}{1 - tz}$$

in view of the Taylor expansion, $-\log(1 - u) = \sum_{n=1}^{\infty} u^n/n$ valid for $|u| < 1$. The integrand has the three singularities $z = 1/t$ (simple pole), $z = \infty$ (double pole) and $z = 0$ (which may be an essential singularity). Accordingly (2.21) becomes

$$\frac{1}{2\pi\sqrt{-1}} \int_C F(z) dz = 0, \quad \text{where}$$

$$F(z) = \frac{z}{1 - tz} \tau_n(x - [t] + [z^{-1}]) \tau_{n+1}(x - [z^{-1}])$$

and the integration contour C is taken along a positively oriented small circle around $z = 0$ such that $z = 1/t, \infty$ are exterior to it. The residues at the two poles are evaluated as

$$\operatorname*{Res}_{z=1/t} F(z) dz = -t^{-2} \tau_n(x) \tau_{n+1}(x - [t]),$$

and

$$\operatorname*{Res}_{z=\infty} F(z) dz = -\operatorname*{Res}_{w=0} F(w^{-1}) w^{-2} dw, \quad \text{where } zw = 1$$

$$= \frac{d}{dw} \left(\frac{1}{t - w} \tau_n(x - [t] + [w]) \tau_{n+1}(x - [w]) \right) \Bigg|_{w=0}$$

$$= \frac{1}{(t - w)^2} \tau_n(x - [t] + [w]) \tau_{n+1}(x - [w])$$

$$+ \frac{1}{w(t - w)} D_{\delta_w} \tau_n(x - [t] + [w]) \cdot \tau_{n+1}(x - [w]) \Bigg|_{w=0}$$

$$= t^{-2} \tau_n(x - [t]) \tau_{n+1}(x) + t^{-1} D_{x_1} \tau_n(x - [t]) \cdot \tau_{n+1}(x).$$

We find through Cauchy's residue theorem that $\operatorname*{Res}_{z=1/t} F(z) dz + \operatorname*{Res}_{z=\infty} F(z) dz = 0$, which implies (2.22a).

For other cases the differential form of Ω and the singularities of $F(z)$ are listed below:

Case	Ω	Singularities of $F(z)$ other than $z = 0$
(b)	$\dfrac{z \, dz}{(1 - tz)^2}$	$z = \infty$ (simple pole), $z = 1/t$ (double pole)
(c)	$\dfrac{z \, dz}{(1 - tz)(1 - sz)}$	$z = 1/t, 1/s, \infty$ (simple poles)
(d)	$\dfrac{z \, dz}{(1 - tz)^2(1 - sz)}$	$z = 1/s$ (simple pole), $z = 1/t$ (double pole)

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We see that (2.21) takes the form $\frac{1}{2\pi\sqrt{-1}}\int_C F(z)\mathrm{d}z = 0$ for each case as well as (a). Here C encircles $z = 0$ so that all the other singularities are exterior to it. Residue calculus again leads to the desired results (2.22b)–(2.22d). □

For later convenience, we summarize how to derive an infinite series of Hirota differential equations from the functional relation

$$\sum_{i+j=-1-d} X_i^- f \otimes X_j^+ g = 0 \quad \text{for a given } d \in \mathbb{Z}, \tag{2.24}$$

where $f \otimes g = f(\boldsymbol{x}')g(\boldsymbol{x})$ is regarded as an element of $\mathbb{C}[\boldsymbol{x}'] \otimes \mathbb{C}[\boldsymbol{x}]$. Now let us consider the Taylor expansion of (2.24) at $\boldsymbol{x}' = \boldsymbol{x}$, i.e., replace $(\boldsymbol{x}', \boldsymbol{x})$ with $(\boldsymbol{x} + \boldsymbol{u}, \boldsymbol{x} - \boldsymbol{u})$ and then expand with respect to the variables $\boldsymbol{u} = (u_1, u_2, \ldots)$. We thus obtain

$$\frac{1}{2\pi\sqrt{-1}} \oint z^d \mathrm{d}z\, e^{-2\xi(\boldsymbol{u},z)} e^{\xi(\tilde{\partial}_{\boldsymbol{u}},z^{-1})} f(\boldsymbol{x}+\boldsymbol{u})g(\boldsymbol{x}-\boldsymbol{u}) = 0,$$

thereby,

$$\sum_{i+j+d=-1} p_i(-2\boldsymbol{u})p_{-j}(\tilde{\partial}_{\boldsymbol{u}})f(\boldsymbol{x}+\boldsymbol{u})g(\boldsymbol{x}-\boldsymbol{u}) = 0.$$

If we remember the definition of the Hirota differential (2.8), we can verify that

$$p_{-j}(\tilde{\partial}_{\boldsymbol{u}})f(\boldsymbol{x}+\boldsymbol{u})g(\boldsymbol{x}-\boldsymbol{u}) = p_{-j}(\tilde{D}_{\boldsymbol{x}})e^{\sum_{n=1}^{\infty} u_n D_{x_n}} f(\boldsymbol{x}) \cdot g(\boldsymbol{x}),$$

with $\tilde{D}_{\boldsymbol{x}}$ denoting $(D_{x_1}, D_{x_2}/2, D_{x_3}/3, \ldots)$. Introduce the generating function

$$G_d(f(\boldsymbol{x}), g(\boldsymbol{x}); \boldsymbol{u}) := \sum_{i+j+d=-1} p_i(-2\boldsymbol{u})p_{-j}(\tilde{D}_{\boldsymbol{x}})e^{\sum_{n=1}^{\infty} u_n D_{x_n}} f(\boldsymbol{x}) \cdot g(\boldsymbol{x}) \tag{2.25}$$

in $\boldsymbol{u} = (u_1, u_2, \ldots)$. Therefore we conclude that

$$G_d(f(\boldsymbol{x}), g(\boldsymbol{x}); \boldsymbol{u}) = 0. \tag{2.26}$$

Example 2.8. Let us write down a few of Hirota differential equations arising from the mKP hierarchy (2.12). Put $d = 1$, $f = \tau_n$, and $g = \tau_{n+1}$ in (2.26). Then, the coefficients of $1 = \boldsymbol{u}^0$ and u_1 give

$$\left(D_{x_1}^2 + D_{x_2}\right)\tau_n \cdot \tau_{n+1} = 0 \tag{2.27}$$

and

$$\left(D_{x_1}^3 - 3D_{x_1}D_{x_2} - 4D_{x_3}\right)\tau_n \cdot \tau_{n+1} = 0, \tag{2.28}$$

respectively.

3 From KP hierarchy to Painlevé III chain

In this section, we derive a system of nonlinear ordinary differential equations from the ℓ-periodic mKP hierarchy ($\ell \geq 2$) through a homogeneity constraint. This system, called the P_{III}-*chain*, provides a higher order generalization of the third Painlevé equation, which indeed coincides with the original one when $\ell = 2$.

3.1 Similarity reduction

Let $\tau_n(\boldsymbol{x})$ be a solution of the mKP hierarchy, (2.12) or (2.21), satisfying the ℓ-periodic condition: $\tau_{n+\ell} = \tau_n$ ($\ell \geq 2$), and the similarity condition:

$$E\tau_n(\boldsymbol{x}) = d_n\tau_n(\boldsymbol{x}) \quad (d_n \in \mathbb{C}) \tag{3.1}$$

where $E = \sum_{n=1}^{\infty} nx_n\partial/\partial x_n$. Introduce the functions $\sigma_n = \sigma_n(a, s)$ ($n \in \mathbb{Z}/\ell\mathbb{Z}$) defined by $\sigma_n(a, s) = \tau_n(\boldsymbol{x})$ under the substitution

$$x_1 = s + a \quad \text{and} \quad x_n = \frac{a}{n} \quad (n \geq 2). \tag{3.2}$$

Note that s will play a role of the independent variable while a will be regarded as a constant parameter.

Proposition 3.1. *The functions $\sigma_n = \sigma_n(a, s)$ satisfy the system of bilinear equations*

$$(D_s + 1)\,\sigma_n(a - 1, s) \cdot \sigma_{n+1}(a, s) - \sigma_n(a, s)\sigma_{n+1}(a - 1, s) = 0, \tag{3.3}$$

$$(sD_s + a + d_{n+1} - d_n)\,\sigma_n(a, s) \cdot \sigma_{n+1}(a, s) - a\sigma_n(a - 1, s)\sigma_{n+1}(a + 1, s) = 0. \tag{3.4}$$

Proof. Observe that

$$\frac{\mathrm{d}}{\mathrm{d}s} = \sum_{n=1}^{\infty} \frac{\mathrm{d}x_n}{\mathrm{d}s}\frac{\partial}{\partial x_n} = \frac{\partial}{\partial x_1}. \tag{3.5}$$

Also we see that

$$E = \sum_{n=1}^{\infty} nx_n\frac{\partial}{\partial x_n} = s\frac{\partial}{\partial x_1} + a\sum_{n=1}^{\infty}\frac{\partial}{\partial x_n} = s\frac{\mathrm{d}}{\mathrm{d}s} + a\delta_1. \tag{3.6}$$

Hence, with the aid of the homogeneity (3.1), we can readily verify (3.3) and (3.4) from (2.22a) and (2.22b), respectively. $\qquad\square$

For appropriately chosen dependent variables, we will derive a system of nonlinear ordinary differential equations from the bilinear equations in Prop. 3.1.

Theorem 3.2. *Define the functions $f_n = f_n(a, s)$ and $g_n = g_n(a, s)$ ($n \in \mathbb{Z}/\ell\mathbb{Z}$) by*

$$f_n(a, s) = \frac{\sigma_{n-1}(a, s)\sigma_n(a - 1, s)}{\sigma_{n-1}(a - 1, s)\sigma_n(a, s)}, \quad g_n(a, s) = \frac{\sigma_{n-1}(a - 1, s)\sigma_n(a + 1, s)}{\sigma_{n-1}(a, s)\sigma_n(a, s)}. \tag{3.7}$$

Then these functions satisfy the system of ordinary differential equations

$$s\frac{\mathrm{d}f_n}{\mathrm{d}s} = (s(f_{n+1} - f_n) + a(g_n - g_{n+1}) + \alpha_n - 1)\,f_n, \tag{3.8a}$$

$$\frac{\mathrm{d}g_n}{\mathrm{d}s} = f_ng_n - f_{n-1}g_{n-1}, \tag{3.8b}$$

where $\alpha_n = d_{n-1} - 2d_n + d_{n+1} + 1$.

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Proof. To simplify an expression we write as $\overline{\sigma}_n = \sigma_n(a+1, s)$ and $\underline{\sigma}_n = \sigma_n(a-1, s)$, while $\sigma_n = \sigma_n(a, s)$, for brevity. It follows respectively from (3.3) and (3.4) that

$$\frac{D_s \underline{\sigma}_{n-1} \cdot \sigma_n}{\underline{\sigma}_{n-1} \sigma_n} = f_n - 1, \tag{3.9}$$

$$\frac{s D_s \sigma_{n-1} \cdot \sigma_n}{\sigma_{n-1} \sigma_n} = a g_n - a - d_n + d_{n-1}. \tag{3.10}$$

Therefore the logarithmic derivative of f_n reads

$$\frac{1}{f_n} \frac{\mathrm{d} f_n}{\mathrm{d} s} = \frac{\sigma_{n-1}{}'}{\sigma_{n-1}} + \frac{\underline{\sigma}_n{}'}{\underline{\sigma}_n} - \frac{\underline{\sigma}_{n-1}{}'}{\underline{\sigma}_{n-1}} - \frac{\sigma_n{}'}{\sigma_n}, \qquad \text{where} \quad {}' = \frac{\mathrm{d}}{\mathrm{d}s}$$

$$= -\frac{D_s \underline{\sigma}_{n-1} \cdot \sigma_n}{\underline{\sigma}_{n-1} \sigma_n} + \frac{D_s \underline{\sigma}_n \cdot \sigma_{n+1}}{\underline{\sigma}_n \sigma_{n+1}} + \frac{D_s \sigma_{n-1} \cdot \sigma_n}{\sigma_{n-1} \sigma_n} - \frac{D_s \sigma_n \cdot \sigma_{n+1}}{\sigma_n \sigma_{n+1}}$$

$$= f_{n+1} - f_n + \frac{a(g_n - g_{n+1}) + d_{n-1} - 2d_n + d_{n+1}}{s},$$

which is exactly (3.8a). Likewise we have

$$\frac{1}{g_n} \frac{\mathrm{d} g_n}{\mathrm{d} s} = \frac{D_s \underline{\sigma}_{n-1} \cdot \sigma_n}{\underline{\sigma}_{n-1} \sigma_n} - \frac{D_s \sigma_{n-1} \cdot \overline{\sigma}_n}{\sigma_{n-1} \overline{\sigma}_n} = f_n - \overline{f}_n.$$

If we take into account that (see (3.7))

$$\overline{f}_n = \frac{f_{n-1} g_{n-1}}{g_n}, \tag{3.11}$$

then we arrive at (3.8b). □

One can find that (3.8) possesses two conserved quantities

$$\prod_{n=1}^{\ell} f_n = 1 \quad \text{and} \quad \sum_{n=1}^{\ell} g_n = \ell, \tag{3.12}$$

and thus it is essentially of $2(\ell - 1)$th order. If $\ell = 2$ it is indeed equivalent to the third Painlevé equation P_{III}, which is obviously of second order, as demonstrated in the next subsection. For this reason we call (3.8) the P_{III}-*chain.*

Remark 3.3. The P_{III}-chain has been investigated in a quite different manner by Adler et al. in the study of Darboux transformations of a sequence of Schrödinger operators with quadratic eigenvalue dependence. It is closely related to the relativistic Toda lattice equation ([38]). See an excellent review article [4], which involves the case of other Painlevé equations also.

3.2 Example: two-periodic case and P_{III}

Consider the case where $\ell = 2$. Introduce the canonical variables $q(s)$ and $p(s)$ as

$$q = f_1 = \frac{\sigma_0 \underline{\sigma}_1}{\underline{\sigma}_0 \sigma_1}, \quad p = -a f_0 g_0 = -a \frac{\underline{\sigma}_0 \overline{\sigma}_0}{\sigma_0{}^2}.$$

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Thus (3.8) is rewritten into the Hamiltonian system

$$\frac{\mathrm{d}q}{\mathrm{d}s} = \frac{\partial H}{\partial p}, \quad \frac{\mathrm{d}p}{\mathrm{d}s} = -\frac{\partial H}{\partial q},$$

with the Hamiltonian function $H = H(q, p; s)$ defined by

$$sH = q^2 p^2 - \left(sq^2 - 2(a + d_0 - d_1)q - s \right) p - 2asq.$$

This is identical to the Hamiltonian form of P_{III}; see [36].

Alternatively, we can derive P_{III} more directly from the bilinear equations for τ-functions. First we differentiate with respect to t the equation (2.22a) after shifting the variables x to $x + [t]/2$; thus, we have

$$(tD_{x_1}D_{\delta_t} - 2tD_{x_1} + D_{\delta_t})\,\tau_n\left(x - \frac{[t]}{2}\right) \cdot \tau_{n+1}\left(x + \frac{[t]}{2}\right) + D_{\delta_t}\tau_n\left(x + \frac{[t]}{2}\right) \cdot \tau_{n+1}\left(x - \frac{[t]}{2}\right) = 0. \quad (3.13)$$

Applying change of variables (3.2) and $t = 1$ together with the homogeneity (3.1), we have from (3.13) that

$$\left(\frac{1}{s}\mathcal{D}^2 + (s + 2a + d_{n+1} - d_n - 1)D_s + d_{n+1} - d_n \right) \sigma_n(a - 1, s) \cdot \sigma_{n+1}(a, s)$$

$$+ (\mathcal{D} + d_{n+1} - d_n)\,\sigma_n(a, s) \cdot \sigma_{n+1}(a - 1, s) = 0, \quad (3.14)$$

where the calligraphic symbol $\mathcal{D}$ stands for the Hirota differential with respect to $s\mathrm{d}/\mathrm{d}s$. Note that (3.14) is still valid without requiring the two-periodicity. We are now interested in a second order differential equation satisfied by the variable $q = q(s)$. One can verify, only by using definition of the Hirota differential, the formula

$$\left(s\frac{\mathrm{d}}{\mathrm{d}s} \right)^2 \log q = \frac{\mathcal{D}^2\sigma_0 \cdot \sigma_1}{\sigma_0 \underline{\sigma}_1} - \frac{\mathcal{D}^2\sigma_1 \cdot \sigma_0}{\sigma_1 \underline{\sigma}_0} - \left(\frac{\mathcal{D}\sigma_0 \cdot \sigma_1}{\sigma_0 \underline{\sigma}_1} \right)^2 + \left(\frac{\mathcal{D}\sigma_1 \cdot \sigma_0}{\sigma_1 \underline{\sigma}_0} \right)^2. \quad (3.15)$$

On the other hand, we see from (3.4) and (3.14) in view of $\sigma_{n+2} = \sigma_n$ that

$$\frac{\mathcal{D}\sigma_0 \cdot \sigma_1}{\sigma_0 \underline{\sigma}_1} = s\left(1 - \frac{1}{q} \right), \quad \frac{\mathcal{D}\sigma_1 \cdot \sigma_0}{\sigma_1 \underline{\sigma}_0} = s(1 - q),$$

$$\frac{\mathcal{D}^2\sigma_0 \cdot \sigma_1}{\sigma_0 \underline{\sigma}_1} = s\left(2s + 2a - 1 - (2s + 2a + 2d_0 - 2d_1 - 1)\frac{1}{q} \right),$$

$$\frac{\mathcal{D}^2\sigma_1 \cdot \sigma_0}{\sigma_1 \underline{\sigma}_0} = s\left(2s + 2a - 1 - (2s + 2a + 2d_1 - 2d_0 - 1)q \right).$$

Substituting these into (3.15), we finally arrive at

$$\left(s\frac{\mathrm{d}}{\mathrm{d}s} \right)^2 \log q = s^2\left(q^2 - \frac{1}{q^2} \right) + s\left((2a + 2d_1 - 2d_0 - 1)q - (2a + 2d_0 - 2d_1 - 1)\frac{1}{q} \right),$$

which is equivalent to the standard form of P_{III}:

$$\frac{\mathrm{d}^2 q}{\mathrm{d}s^2} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}s} \right)^2 - \frac{1}{s}\frac{\mathrm{d}q}{\mathrm{d}s} + \frac{1}{s}\left(\alpha q^2 + \beta \right) + \gamma q^3 + \frac{\delta}{q}, \quad (3.16)$$

where $\alpha = 2a + 2d_1 - 2d_0 - 1$, $\beta = -2a + 2d_1 - 2d_0 + 1$, and $\gamma = -\delta = 1$.

Remark 3.4. To be more precise, the third Painlevé equation should be divided into three types, i.e., $D_6^{(1)}$, $D_7^{(1)}$, and $D_8^{(1)}$, from the viewpoint of space of initial conditions; see [34, 39]. In this context our case (3.16) is classified as the $D_6^{(1)}$ type. Recall also that the $D_8^{(1)}$ type is algebraically equivalent to the $D_6^{(1)}$ one with special value of parameters; see [52]. It still remains open how to derive the $D_7^{(1)}$ type from an integrable system such as the KP hierarchy.

3.3 Affine Weyl group symmetry

Now we shall concern birational symmetries of the P_{III}-chain, (3.8). Here, to be precise, a birational transformation of the variables (f_n, g_n, s) is said to be a symmetry of the P_{III}-chain if it keeps the system invariant except changing the constant parameters (a, α_n) involved.

First, we consider a symmetry shifting the parameter a to $a \pm 1$. Write as $\overline{f}_n = f_n(a+1, s)$ while $f_n = f_n(a, s)$ and so on. Then we have (3.11): $\overline{f}_n = f_{n-1}g_{n-1}/g_n$, and

$$
\begin{aligned}
\overline{g}_n &= \frac{\sigma_{n-1}(a, s)\sigma_n(a+2, s)}{\sigma_{n-1}(a+1, s)\sigma_n(a+1, s)} \\
&= \frac{1}{a+1}\left(s\left(\log \frac{\overline{\sigma}_{n-1}}{\overline{\sigma}_n}\right)' + a + d_n - d_{n-1} + 1\right), \quad \text{where } ' = \frac{\mathrm{d}}{\mathrm{d}s} \text{ and using (3.4)} \\
&= \frac{1}{a+1}\left(s\left(\log \frac{\overline{\sigma}_{n-1}}{\sigma_{n-2}} + \log \frac{\sigma_{n-2}}{\sigma_{n-1}} + \log \frac{\sigma_{n-1}}{\overline{\sigma}_n}\right)' + a + d_n - d_{n-1} + 1\right) \\
&= \frac{1}{a+1}\left(s\frac{\overline{\sigma}_{n-1}\sigma_n}{\sigma_{n-1}\overline{\sigma}_n} - s\frac{\overline{\sigma}_{n-2}\sigma_{n-1}}{\sigma_{n-2}\overline{\sigma}_{n-1}} + a\frac{\sigma_{n-2}\overline{\sigma}_{n-1}}{\sigma_{n-2}\sigma_{n-1}} + \alpha_{n-1}\right), \quad \text{using (3.3) and (3.4)} \\
&= \frac{1}{a+1}\left(s\left(\overline{f}_n - \overline{f}_{n-1}\right) + ag_{n-1} + \alpha_{n-1}\right).
\end{aligned}
$$

This transformation $(f_n, g_n) \mapsto \left(\overline{f}_n, \overline{g}_n\right)$ is birational and indeed keeps (3.8) invariant except shifting a to $a+1$.

Next, we observe that the system (3.3) and (3.4) of bilinear equations is invariant under the transformation $\iota : (\sigma_n(a, s), s, a, d_n) \mapsto (\sigma_{-n}(-a, s), -s, -a, d_{-n})$. This trivial symmetry can be lifted to a birational symmetry of the P_{III}-chain.

We shall now translate the Weyl group action $\langle r_i \rangle$ of type A, discussed in Sect 2.2, into birational transformations of f_n and g_n. For each $i \in \mathbb{Z}/\ell\mathbb{Z}$, we have that

$$
\tau_{i-1}(x - [t])\tau_{i+1}(x + [t]) = \frac{1}{t}D_{\delta_t}\hat{\tau}_i(x) \cdot \tau_i(x), \tag{3.17}
$$

$$
\tau_{i-1}(x)\tau_{i+1}(x) = D_{x_1}\hat{\tau}_i(x) \cdot \tau_i(x), \tag{3.18}
$$

which can be deduced from (2.19a) by taking the variables as $x - x' = 2[t]$ and as $x = x'$, respectively. If we take account of the reduction condition (3.1) and (3.2), we obtain

$$
a\underline{\sigma}_{i-1}\overline{\sigma}_{i+1} = (\alpha_i - sD_s)\hat{\sigma}_i \cdot \sigma_i,
$$

$$
\sigma_{i-1}\sigma_{i+1} = D_s\hat{\sigma}_i \cdot \sigma_i,
$$

and thereby

$$
\alpha_i\hat{\sigma}_i\sigma_i = s\sigma_{i-1}\sigma_{i+1} + a\underline{\sigma}_{i-1}\overline{\sigma}_{i+1}.
$$

Namely we have

$$
r_i(\sigma_n) = \begin{cases} \dfrac{s\sigma_{i-1}\sigma_{i+1} + a\underline{\sigma}_{i-1}\overline{\sigma}_{i+1}}{\alpha_i\sigma_i} & (n = i), \\[2mm] \sigma_n & (n \neq i). \end{cases}
$$

It is easy to derive the associated transformation on (f_n, g_n).

Finally, it is obvious that a cyclic permutation of the suffixes, $\pi : (f_n, g_n, \alpha_n) \mapsto (f_{n+1}, g_{n+1}, \alpha_{n+1})$, preserves (3.8) invariant.

Summarizing above we arrive at the

Theorem 3.5. *The P_{III}-chain (3.8) is invariant under birational transformations T_a, ι, r_i $(i \in \mathbb{Z}/\ell\mathbb{Z})$, and π defined by*

	Action on a, α_n, and s	Action on f_n	Action on g_n
T_a	$a \mapsto a + 1$	$f_n \mapsto \overline{f}_n$	$g_n \mapsto \overline{g}_n$
ι	$a \mapsto -a, \alpha_n \mapsto \alpha_{-n}, s \mapsto -s$	$f_n \mapsto f_{-n+1}$	$g_n \mapsto g_{-n+1}$
r_i	$\alpha_i \mapsto -\alpha_i,$	$f_i \mapsto f_i - \dfrac{\alpha_i f_i}{sf_i + ag_{i+1}},$	$g_i \mapsto g_i + \dfrac{\alpha_i g_{i+1}}{sf_i + ag_{i+1}},$
	$\alpha_{i\pm1} \mapsto \alpha_{i\pm1} + \alpha_i$	$f_{i+1} \mapsto f_{i+1} + \dfrac{\alpha_i f_{i+1}}{sf_i + ag_{i+1} - \alpha_i}$	$g_{i+1} \mapsto g_{i+1} - \dfrac{\alpha_i g_{i+1}}{sf_i + ag_{i+1}}$
π	$\alpha_n \mapsto \alpha_{n+1}$	$f_n \mapsto f_{n+1}$	$g_n \mapsto g_{n+1}$

Here $\overline{f}_n = f_{n-1}g_{n-1}/g_n$ and $\overline{g}_n = \left(s\left(\overline{f}_n - \overline{f}_{n-1}\right) + ag_{n-1} + \alpha_{n-1}\right)/(a + 1)$.

As indicated in Sect. 2.2, we see that $\langle r_i \ (i \in \mathbb{Z}/\ell\mathbb{Z})\rangle$ provides a realization of an affine Weyl group of type $A_{\ell-1}^{(1)}$, denoted by $W(A_{\ell-1}^{(1)})$. In addition π realizes its rotational Dynkin automorphism. Note that the transformations r_i and π have already appeared in [5]. These symmetries are clearly understood from the view point of the KP hierarchy. However, the nature of T_a and ι seems mysterious from this point and, in the first place, it is still an open problem to determine the group of birational symmetries of the P_{III}-chain.

Remark 3.6 (Toda equation). We shall derive a Toda equation satisfied by a sequence of τ-functions associated with the translation symmetry T_a. First, it follows from (3.14) that

$$
\frac{1}{s}\frac{\mathcal{D}^2\underline{\sigma}_{n-1} \cdot \sigma_n}{\underline{\sigma}_{n-1}\sigma_n} = d_{n-1} - d_n - (s + 2a + d_n - d_{n-1} - 1)\frac{D_s\underline{\sigma}_{n-1} \cdot \sigma_n}{\underline{\sigma}_{n-1}\sigma_n}
$$
$$
+ (d_{n-1} - d_n)\frac{\sigma_{n-1}\underline{\sigma}_n}{\underline{\sigma}_{n-1}\sigma_n} - \frac{\mathcal{D}\sigma_{n-1} \cdot \underline{\sigma}_n}{\underline{\sigma}_{n-1}\sigma_n}
$$
$$
= s + 2a - 1 - (s + 2a + 2d_n - 2d_{n-1} - 1)f_n - \frac{\mathcal{D}\sigma_{n-1} \cdot \underline{\sigma}_n}{\sigma_{n-1}\underline{\sigma}_n}f_n \tag{3.19}
$$

by the use of (3.3) and (3.7). Next, we express (3.3) in the form

$$
\frac{\mathcal{D}\underline{\sigma}_{n-1} \cdot \sigma_n}{\underline{\sigma}_{n-1}\sigma_n} = -s + s\frac{\sigma_{n-1}\underline{\sigma}_n}{\underline{\sigma}_{n-1}\sigma_n}.
$$

By differentiating this with respect to s, we have

$$
\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathcal{D}\underline{\sigma}_{n-1} \cdot \sigma_n}{\underline{\sigma}_{n-1}\sigma_n}\right) = -1 + \left(1 + \frac{\mathcal{D}\sigma_{n-1} \cdot \sigma_n}{\sigma_{n-1}\sigma_n} - \frac{\mathcal{D}\underline{\sigma}_{n-1} \cdot \underline{\sigma}_n}{\underline{\sigma}_{n-1}\underline{\sigma}_n}\right)f_n. \tag{3.20}
$$

Combining (3.19) with (3.20), we obtain

$$\frac{1}{s}\frac{\mathcal{D}^2\underline{\sigma}_{n-1}\cdot\sigma_n}{\underline{\sigma}_{n-1}\sigma_n} - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathcal{D}\underline{\sigma}_{n-1}\cdot\sigma_n}{\underline{\sigma}_{n-1}\sigma_n}\right) = s + 2a - (s + 2a + 2d_n - 2d_{n-1})f_n$$

$$+ \left(\frac{\mathcal{D}\underline{\sigma}_{n-1}\cdot\sigma_n}{\underline{\sigma}_{n-1}\sigma_n} - 2\frac{\mathcal{D}\sigma_{n-1}\cdot\sigma_n}{\sigma_{n-1}\sigma_n}\right)f_n$$

$$= 2a - 2af_n g_n + s(f_n - 1)^2$$

via (3.3) and (3.4). Hence, if we remember (3.9), then we find that

$$\frac{1}{s}\frac{\mathcal{D}^2\sigma_n\cdot\sigma_n}{\sigma_n{}^2} = \frac{1}{s}\frac{\mathcal{D}^2\underline{\sigma}_{n-1}\cdot\sigma_n}{\underline{\sigma}_{n-1}\sigma_n} - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathcal{D}\underline{\sigma}_{n-1}\cdot\sigma_n}{\underline{\sigma}_{n-1}\sigma_n}\right) - \frac{1}{s}\left(\frac{\mathcal{D}\underline{\sigma}_{n-1}\cdot\sigma_n}{\underline{\sigma}_{n-1}\sigma_n}\right)^2$$

$$= 2a - 2af_n g_n$$

$$= 2a - 2a\frac{\sigma_n\overline{\sigma}_n}{\sigma_n{}^2}.$$

Thus, we finally arrive at the *Toda equation*:

$$\frac{1}{2as}\mathcal{D}^2\sigma_n\cdot\sigma_n = \sigma_n{}^2 - \underline{\sigma}_n\overline{\sigma}_n. \tag{3.21}$$

Recall that for the case $\ell = 2$ (P_{III}) such a differential-difference equation of Toda-type has been studied in [36].

3.4 Rational solutions in terms of Schur functions

As previously seen in Sect. 3.1, the P_{III}-chain is by nature equivalent to a similarity reduction of the mKP hierarchy. Also, the mKP hierarchy admits the Schur functions as its homogeneous polynomial solutions; see Example 2.5. Consequently we can construct a particular solution of the P_{III}-chain in terms of the Schur function.

To state the result precisely, we first recall some terminology. A subset $\mathbf{m} \subset \mathbb{Z}$ is said to be a *Maya diagram* if $i \in \mathbf{m}$ (for $i \ll 0$) and $i \notin \mathbf{m}$ (for $i \gg 0$). Each Maya diagram $\mathbf{m} = \{\ldots, m_3, m_2, m_1\}$ ($m_{i+1} < m_i$) corresponds to a unique partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $m_i - m_{i+1} = \lambda_i - \lambda_{i+1} + 1$. For a sequence of integers $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathbb{Z}^\ell$, we associate a Maya diagram

$$\mathbf{m}(\boldsymbol{\nu}) = (\ell\mathbb{Z}_{<\nu_1} + 1) \cup (\ell\mathbb{Z}_{<\nu_2} + 2) \cup \cdots \cup (\ell\mathbb{Z}_{<\nu_\ell} + \ell),$$

and denote by $\lambda(\boldsymbol{\nu})$ its corresponding partition. Note that $\lambda(\boldsymbol{\nu} + \mathbf{1}) = \lambda(\boldsymbol{\nu})$ where $\mathbf{1} = (1, 1, \ldots, 1)$. We call a partition of the form $\lambda(\boldsymbol{\nu})$ an ℓ-*core* partition. A partition λ is ℓ-core if and only if λ has no hook with length of a multiple of ℓ. For example, if $\ell = 3$ and $\boldsymbol{\nu} = (2, 0, 3)$ then the resulting partition reads $\lambda(\boldsymbol{\nu}) = (4, 2, 1, 1)$. Next, we prepare a cyclic chain of the Schur functions attached to ℓ-core partitions that is connected by successive action of vertex operators.

Lemma 3.7. *It holds that*

$$X^+_{\ell\nu_i - |\boldsymbol{\nu}|}S_{\lambda(\boldsymbol{\nu}(i-1))}(\boldsymbol{x}) = \pm S_{\lambda(\boldsymbol{\nu}(i))}(\boldsymbol{x})$$

for arbitrary $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathbb{Z}^\ell$, *where* $\boldsymbol{\nu}(i) = \boldsymbol{\nu} + (\overbrace{1, \ldots, 1}^{i}, \overbrace{0, \ldots, 0}^{\ell-i})$ *and* $|\boldsymbol{\nu}| = \nu_1 + \nu_2 + \cdots + \nu_\ell$.

Finally, we are led to the following expression of rational solutions of the P_{III}-chain in terms of the Schur functions attached to the ℓ-core partitions.

Theorem 3.8. *For any $v \in \mathbb{Z}^\ell$, let*

$$\sigma_n(a, s) = S_{\lambda(v(n))}\left(s + a, \frac{a}{2}, \frac{a}{3}, \frac{a}{4}, \ldots\right).$$

Then $\sigma_n(a, s)$ solve the system of bilinear equations (3.3) and (3.4) when $d_{n+1} - d_n = \ell v_{n+1} - |v|$. Consequently, the 2ℓ-tuple of functions

$$f_n = \frac{\sigma_{n-1}(a, s)\sigma_n(a - 1, s)}{\sigma_{n-1}(a - 1, s)\sigma_n(a, s)}, \quad g_n = \frac{\sigma_{n-1}(a - 1, s)\sigma_n(a + 1, s)}{\sigma_{n-1}(a, s)\sigma_n(a, s)},$$

gives a rational solution of the P_{III}-chain, (3.8), with the parameters $\alpha_n = \ell(v_{n+1} - v_n) + 1$.

Remark 3.9. For the case $\ell = 2$ (P_{III}) the above rational solution has been studied in a different way; cf. [17].

Example 3.10. Consider the polynomial

$$R_\lambda(a, s) = \left(\prod_{(i,j)\in\lambda} h(i, j)\right) S_\lambda\left(s + a, \frac{a}{2}, \frac{a}{3}, \frac{a}{4}, \ldots\right)$$

for a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, where $h(i, j)$ denotes the *hook-length*, $h(i, j) = \lambda_i + {}^{\mathrm{T}}\lambda_j - i - j + 1$, with ${}^{\mathrm{T}}\lambda$ being the transpose (or conjugate) of λ. Interestingly enough, $R_\lambda(a, s)$ comes out to be a polynomial with integer coefficients and monic with respect to both s and a. We give below some examples of $R_\lambda(a, s)$ of small degree:

$$R_\emptyset = 1, \quad R_\square = s + a, \quad R_{\square\square} = s^2 + 2as + a(a + 1),$$
$$R_{\square\square\square} = s^3 + 3as^2 + 3a(a + 1)s + a(a + 1)(a + 2),$$
$$R_{\square\square} = s^3 + 3as^2 + 3a^2 s + (a - 1)a(a + 1),$$
$$R_{\square\square\square\square} = s^4 + 4as^3 + 6a(a + 1)s^2 + 4a(a + 1)(a + 2)s + a(a + 1)(a + 2)(a + 3),$$
$$R_{\square\square\square} = s^4 + 4as^3 + 2a(3a + 1)s^2 + 4a^2(a + 1)s + (a - 1)a(a + 1)(a + 2),$$
$$R_{\square\square} = s^4 + 4as^3 + 6a^2 s^2 + 4(a - 1)a(a + 1)s + (a - 1)a^2(a + 1).$$

This polynomial can be regarded as a generalization of the Umemura polynomial associated with P_{III}; cf. [29].

3.5 Lax formalism

We introduce the *wave function*:

$$\psi_n(\mathbf{x}, k) = \frac{\tau_n(\mathbf{x} - [k^{-1}])}{\tau_n(\mathbf{x})} e^{\xi(\mathbf{x}, k)}, \tag{3.22}$$

which is a function in $\mathbf{x} = (x_1, x_2, \ldots)$ equipped with an additional parameter k (the *spectral variable*). In terms of the wave functions, the linear equations associated with the (modified) KP

hierarchy can be generated from the bilinear identities (recall (2.12) and also Sect. 2.3) in a standard manner; cf. [6, 27]. In what follows, we show that these linear equations naturally induce an auxiliary linear problem, through the similarity reduction, whose integrability condition amounts to the P_{III}-chain (Lax formalism).

We first prepare a key lemma below.

Lemma 3.11. *If $\tau_n = \tau_n(\boldsymbol{x})$ obeys the homogeneity $E\tau_n = d_n\tau_n$. Then it holds that*

$$\left(E - k\frac{\partial}{\partial k}\right)\psi_n = 0. \tag{3.23}$$

Proof. The homogeneity tells us that $(E - k\partial/\partial k)\,\tau_n(\boldsymbol{x} - [k^{-1}]) = d_n\tau_n(\boldsymbol{x} - [k^{-1}])$. Therefore,

$$\left(E - k\frac{\partial}{\partial k}\right)\frac{\tau_n(\boldsymbol{x} - [k^{-1}])}{\tau_n(\boldsymbol{x})} = 0.$$

Combining this with the formula $(E - k\partial/\partial k)\,e^{\xi(\boldsymbol{x},k)} = 0$, we conclude (3.23). $\qquad\square$

Now we set

$$\phi_n(a, s, k) = \psi_n(\boldsymbol{x}, k)$$

under the reduction conditions (3.1) and (3.2).

Lemma 3.12. *The wave functions $\phi_n = \phi_n(a) = \phi_n(a, s, k)$ satisfy the following linear equations:*

$$\frac{\partial}{\partial s}\phi_n(a) = \frac{a + d_{n+1} - d_n - ag_{n+1}}{s}\phi_n(a) + k\phi_{n+1}(a), \tag{3.24}$$

$$\phi_n(a - 1) = f_{n+1}\phi_n(a) - k\phi_{n+1}(a), \tag{3.25}$$

$$\left(k\frac{\partial}{\partial k} - s\frac{\partial}{\partial s}\right)\phi_n(a) = a\,(g_{n+1} - 1)\,\phi_n(a) + akg_{n+1}\phi_{n+1}(a + 1). \tag{3.26}$$

Proof. We shall deduce (3.24), (3.25), and (3.26) respectively from (2.22a), (2.22c), and (2.22d) in Lemma 2.7. First, it follows from (2.22a) with t replaced by $1/k$ that

$$\frac{\partial}{\partial x_1}\left(\frac{\tau_n(\boldsymbol{x} - [k^{-1}])}{\tau_n(\boldsymbol{x})}\right) + \left(\frac{D_{x_1}\tau_n(\boldsymbol{x})\cdot\tau_{n+1}(\boldsymbol{x})}{\tau_n(\boldsymbol{x})\tau_{n+1}(\boldsymbol{x})} + k\right)\frac{\tau_n(\boldsymbol{x} - [k^{-1}])}{\tau_n(\boldsymbol{x})} - k\frac{\tau_{n+1}(\boldsymbol{x} - [k^{-1}])}{\tau_{n+1}(\boldsymbol{x})} = 0,$$

i.e.,

$$\frac{\partial\psi_n}{\partial x_1} + \frac{D_{x_1}\tau_n\cdot\tau_{n+1}}{\tau_n\tau_{n+1}}\psi_n - k\psi_{n+1} = 0. \tag{3.27}$$

Remembering (3.5): $\partial/\partial x_1 = \partial/\partial s$, we see that

$$\frac{\partial\phi_n}{\partial s} + \frac{D_s\sigma_n\cdot\sigma_{n+1}}{\sigma_n\sigma_{n+1}}\phi_n - k\phi_{n+1} = 0,$$

which implies (3.24) through (3.10).

Next, (2.22c) with s replaced by $1/k$ shows that

$$\psi_n(\boldsymbol{x} - [t], k) = \frac{\tau_n(\boldsymbol{x})\tau_{n+1}(\boldsymbol{x} - [t])}{\tau_n(\boldsymbol{x} - [t])\tau_{n+1}(\boldsymbol{x})}\psi_n(\boldsymbol{x}, k) - tk\psi_{n+1}(\boldsymbol{x}, k),$$

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which thus yields

$$\phi_n(a - 1) = \frac{\sigma_n(a, s)\sigma_{n+1}(a - 1, s)}{\sigma_n(a - 1, s)\sigma_{n+1}(a, s)}\phi_n(a) - k\phi_{n+1}(a)$$

under (3.2) and $t = 1$. By (3.7) it is immediate to obtain (3.25).

Finally, it follows from (2.22d) with s replaced by $1/k$ that

$$\delta_t\psi_n(\mathbf{x}, k) + \frac{D_{\delta_t}\tau_n(\mathbf{x}) \cdot \tau_{n+1}(\mathbf{x})}{\tau_n(\mathbf{x})\tau_{n+1}(\mathbf{x})}\psi_n(\mathbf{x}, k) - tk\frac{\tau_n(\mathbf{x} - [t])\tau_{n+1}(\mathbf{x} + [t])}{\tau_n(\mathbf{x})\tau_{n+1}(\mathbf{x})}\psi_{n+1}(\mathbf{x} + [t], k) = 0. \quad (3.28)$$

Consider the substitution (3.2) and $t = 1$. Applying Lemma 3.11 together with (3.6), we find that

$$\delta_t\psi_n(\mathbf{x}, k) = \frac{1}{a}\left(E - s\frac{\partial}{\partial s}\right)\phi_n = \frac{1}{a}\left(k\frac{\partial}{\partial k} - s\frac{\partial}{\partial s}\right)\phi_n.$$

Similarly we know that

$$\begin{aligned}
\frac{D_{\delta_1}\sigma_n \cdot \sigma_{n+1}}{\sigma_n\sigma_{n+1}} &= \delta_1 \log \frac{\sigma_n}{\sigma_{n+1}} = \frac{1}{a}\left(E - s\frac{\partial}{\partial s}\right)\log \frac{\sigma_n}{\sigma_{n+1}} \\
&= \frac{1}{a}\left(d_n - d_{n+1} - s\frac{\partial}{\partial s}\right)\log \frac{\sigma_n}{\sigma_{n+1}}, \quad \text{using (3.1)} \\
&= 1 - g_{n+1}, \quad \text{using (3.10).}
\end{aligned}$$

If we put it all together, we then obtain (3.26) from (3.28). $\qquad\square$

The linear equations (3.25) can be solved for $\phi_n(a)$ by virtue of the ℓ-periodicity; i.e.,

$$\phi_n(a) = \frac{1}{1 - k^\ell} \sum_{j=1}^{\ell} k^{j-1}\left(\prod_{i=1}^{j} \frac{1}{f_{n+i}}\right)\phi_{n+j-1}(a - 1).$$

Note here that the suffix n should be regarded suitably as an element of $\mathbb{Z}/\ell\mathbb{Z}$. If we shift a to $a+1$, this expression takes the form

$$\begin{aligned}
\phi_n(a + 1) &= \frac{1}{1 - k^\ell} \sum_{j=1}^{\ell} k^{j-1}\left(\prod_{i=1}^{j} \frac{1}{\overline{f}_{n+i}}\right)\phi_{n+j-1}(a) \\
&= \frac{1}{1 - k^\ell}\frac{1}{g_n} \sum_{j=1}^{\ell} k^{j-1}g_{n+j}\left(\prod_{i=1}^{j} \frac{1}{f_{n+i-1}}\right)\phi_{n+j-1}(a). \quad (3.29)
\end{aligned}$$

Here recall the abbreviated notation $\overline{f}_n := f_n(a + 1, s)$ and the contiguity relation (3.11).

We are now ready to present a system of linear differential equations for the wave functions $\phi_n = \phi_n(a, s, k)$ ($n \in \mathbb{Z}/\ell\mathbb{Z}$), from which the P_{III}-chain (3.8) emerges as its compatibility condition. First, eliminating $\partial\phi_n/\partial s$ and $\phi_n(a + 1)$ from (3.26) by (3.24) and (3.29), we obtain the linear differential equation with respect to the spectral variable k:

$$k\frac{\partial}{\partial k}\phi_n = (d_{n+1} - d_n)\phi_n + sk\phi_{n+1} + \frac{a}{1 - k^\ell} \sum_{j=1}^{\ell} k^j g_{n+j+1}\left(\prod_{i=1}^{j} \frac{1}{f_{n+i}}\right)\phi_{n+j}. \quad (3.30)$$

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This has $\ell+1$ regular singularities at $k = 0$, $\exp\left(2\pi\sqrt{-1}n/\ell\right)$ $(n \in \mathbb{Z}/\ell\mathbb{Z})$ and an irregular singularity at $k = \infty$ whose Poincaré rank equals one. This expression is, however, redundant in some sense; i.e., we can reduce the number of singularities appropriately by changing the variables. Let

$$\Phi = {}^{\mathrm{T}}\left(\phi_0, k\phi_1, \ldots, k^{\ell-1}\phi_{\ell-1}\right) \quad \text{and} \quad z = k^{\ell}.$$

Then (3.30) is converted to the equation

$$\frac{\partial\Phi}{\partial z} = A\Phi = \left(C + \frac{A_0}{z} + \frac{A_1}{z-1}\right)\Phi, \tag{3.31}$$

where the $\ell \times \ell$ matrices C, A_0, and A_1 are given as follows:

$$C = \begin{pmatrix} & O \\ s/\ell & \end{pmatrix},$$

$$A_0 = \begin{pmatrix} e_0 & s/\ell & & & \\ & e_1 & s/\ell & & \\ & & \ddots & \ddots & \\ & & & e_{\ell-2} & s/\ell \\ & & & & e_{\ell-1} \end{pmatrix} + \begin{pmatrix} 0 & v_{0,1} & v_{0,2} & \cdots & v_{0,\ell-1} \\ & 0 & v_{1,2} & \cdots & v_{1,\ell-1} \\ & & \ddots & \ddots & \vdots \\ & & & 0 & v_{\ell-2,\ell-1} \\ & & & & 0 \end{pmatrix},$$

$$A_1 = -(v_{i,j})_{0 \le i,j \le \ell-1}$$

with

$$e_n = \frac{d_{n+1} - d_n + n}{\ell} \quad \text{and} \quad v_{n,n+j} = \frac{ag_{n+j+1}}{\ell}\prod_{i=1}^{j}\frac{1}{f_{n+i}} = \frac{ag_{n+1}}{\ell}\prod_{i=1}^{j}\frac{1}{\overline{f}_{n+i+1}}$$

for $0 \le n \le \ell-1$ and $1 \le j \le \ell$. Note that we read appropriately the suffixes of dependent variables f_i, g_i, and $v_{i,j}$ modulo ℓ. Hence (3.31) has regular singularities at $z = 0, 1$ and an irregular singularity at $z = \infty$. To be more precise, the exponents at $z = 0$ and $z = 1$ turn out to be $(e_0, e_1, \ldots, e_{\ell-1})$ and $(-a, 0, \ldots, 0)$, which equal by definition the eigenvalues of A_0 and A_1, respectively. The latter can be computed in the following manner. Let $\boldsymbol{f}$ and $\boldsymbol{g}$ be the row vectors defined by

$$\boldsymbol{f} = \left(\prod_{i=1}^{j}\frac{1}{\overline{f}_{i+1}}\right)_{0 \le j \le \ell-1} = \left(1, \frac{1}{\overline{f}_2}, \frac{1}{\overline{f}_2\overline{f}_3}, \ldots, \frac{1}{\overline{f}_2\overline{f}_3\cdots\overline{f}_\ell}\right),$$

$$\boldsymbol{g} = \left(g_{j+1}\prod_{i=1}^{j}\overline{f}_{i+1}\right)_{0 \le j \le \ell-1} = \left(g_1, g_2\overline{f}_2, g_3\overline{f}_2\overline{f}_3, \ldots, g_\ell\overline{f}_2\overline{f}_3\cdots\overline{f}_\ell\right).$$

We observe that ${}^{\mathrm{T}}\boldsymbol{g}\cdot\boldsymbol{f} = -(\ell/a)A_1$ and $\boldsymbol{f}\cdot{}^{\mathrm{T}}\boldsymbol{g} = g_1+g_2+\cdots+g_\ell = \ell$ via (3.12), thereby, the exponents at $z = 1$ are evaluated as $(-a, 0, \ldots, 0)$. Further, we say $z = \infty$ to be an irregular singularity of Poincaré rank $1/\ell$ in the sense that (3.31) has a formal matrix solution of the form

$$\Xi = \mathrm{diag}\left(1, z^{1/\ell}, z^{2/\ell}, \ldots, z^{(\ell-1)/\ell}\right) \cdot \left(\zeta^{ij}\right)_{0 \le i,j \le \ell-1} \cdot \left(I + \sum_{i=1}^{\infty}\Xi^{(i)}z^{-i/\ell}\right) \cdot e^{T(z)}$$

near $z = \infty$, where

$$T(z) = -\mathrm{diag}\left(1, \zeta, \zeta^2, \ldots, \zeta^{\ell-1}\right)sz^{1/\ell} - \frac{a}{\ell}\log z$$

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and ζ denotes a primitive ℓth root of unity. Although the series Ξ is in general divergent, it expresses the asymptotic behavior of an actual solution in some sectorial domain around $z = \infty$. We refer to [14] for the general theory of solutions around an irregular singularity.

On the other hand, we obtain from (3.24) the linear differential equation with respect to s:

$$\frac{\partial}{\partial s}\Phi = B\Phi = \left(\begin{pmatrix} u_0 & 1 & & & \\ & u_1 & 1 & & \\ & & \ddots & \ddots & \\ & & & u_{\ell-2} & 1 \\ & & & & u_{\ell-1} \end{pmatrix} + z \begin{pmatrix} & & & \\ & & & \\ & & & \\ 1 & & & \end{pmatrix} \right)\Phi, \tag{3.32}$$

where $u_n = (a + d_{n+1} - d_n - ag_{n+1})/s$.

In the context of monodromy preserving deformations, (3.32) governs a deformation of (3.31) along a parameter s with keeping its monodromy matrices and Stokes multipliers invariant. Finally, we give an alternative characterization of the P_{III}-chain as a compatibility condition of the linear problem.

Theorem 3.13 (Lax representation). *The integrability condition*

$$\left[\frac{\partial}{\partial s} - B, \frac{\partial}{\partial z} - A \right] = 0$$

of the system (3.31) *and* (3.32) *is equivalent to the P_{III}-chain,* (3.8).

Remark 3.14. If $\ell = 2$, the system (3.31) and (3.32) is equivalent to the Lax pair for P_{III} that was recently found in a moduli-theoretical approach [37] (see also [33]).

4 From KP hierarchy to Painlevé IV/V chain

In this section we demonstrate a similarity reduction from the ℓ-periodic mKP hierarchy ($\ell \geq 3$) to the higher order Painlevé equation $P(A_{\ell-1}^{(1)})$ or, equivalently, the Darboux chain with period ℓ, which coincides with the fourth and fifth Painlevé equations when $\ell = 3$ and $\ell = 4$, respectively; cf. [31].

4.1 Similarity reduction

Fix an integer $\ell \geq 3$. Let $\tau_n(\boldsymbol{x})$ be a solution of the mKP hierarchy, (2.12) or (2.21). Suppose it fulfills the following three conditions: (i) homogeneity $E\tau_n = d_n\tau_n$ ($d_n \in \mathbb{C}$); (ii) ℓ-periodicity $\tau_{n+\ell} = \tau_n$; (iii) specialization $x_n = 0$ ($n \neq 1, 2$). Note here that this specialization is intended to eliminate all time evolutions except the first two. In view of (i), we can take x_2 to be a constant without loss of generality. For instance, we fix

$$x_1 = x, \quad x_2 = -\frac{1}{2}, \quad \text{and} \quad x_n = 0 \quad (n \geq 3). \tag{4.1}$$

Under the above constraints, we set $\sigma_n(x) = \tau_n(\boldsymbol{x})$.

Proposition 4.1. *It holds that*

$$\left(D_x{}^2 + xD_x + d_{n+1} - d_n\right)\sigma_n \cdot \sigma_{n+1} = 0. \tag{4.2}$$

Proof. We know from (4.1) that

$$\frac{\partial}{\partial x_1} = \frac{\mathrm{d}}{\mathrm{d}x} \quad \text{and} \quad \frac{\partial}{\partial x_2} = x\frac{\mathrm{d}}{\mathrm{d}x} - E. \tag{4.3}$$

Thus we deduce (4.2) immediately from (2.27). □

Let us consider an ℓ-tuple of functions $w_n = w_n(x)$ ($n \in \mathbb{Z}/\ell\mathbb{Z}$) defined by

$$w_n = \frac{x}{2} + \frac{D_x\sigma_n \cdot \sigma_{n+1}}{\sigma_n\sigma_{n+1}}. \tag{4.4}$$

By virtue of Prop. 4.1, we have then the

Theorem 4.2. *It holds that*

$$\frac{\mathrm{d}}{\mathrm{d}x}(w_n + w_{n-1}) = w_n{}^2 - w_{n-1}{}^2 + \alpha_n \quad (n \in \mathbb{Z}/\ell\mathbb{Z}) \tag{4.5}$$

where $\alpha_n = d_{n-1} - 2d_n + d_{n+1} + 1$.

The sequence (4.5) of ordinary differential equations with quadratic nonlinearity is known as the *Darboux chain* (with period ℓ). It emerged originally from the spectral theory of Schrödinger operators in connection with Darboux transformations. Moreover it provides a higher order generalization of P_{IV} and P_{V} which correspond to the cases $\ell = 3$ and 4, respectively. For details to [3, 53].

Remark 4.3. If we consider the change of variables as (cf. [45])

$$f_n = w_n + w_{n-1} = x + \frac{D_x\sigma_{n-1} \cdot \sigma_{n+1}}{\sigma_{n-1}\sigma_{n+1}},$$

then the Darboux chain (4.5) takes the following expression:
(i) if $\ell = 2g + 1$ ($g = 1, 2, \ldots$),

$$P(A_{2g}^{(1)}) : \qquad \frac{\mathrm{d}f_n}{\mathrm{d}x} = f_n\left(\sum_{i=1}^g f_{n+2i-1} - \sum_{i=1}^g f_{n+2i}\right) + \alpha_n;$$

(ii) if $\ell = 2g + 2$ ($g = 1, 2, \ldots$),

$$P(A_{2g+1}^{(1)}) : \qquad \frac{\ell x}{2}\frac{\mathrm{d}f_n}{\mathrm{d}x} = f_n\left(\sum_{1 \le i \le j \le g} f_{n+2i-1}f_{n+2j} - \sum_{1 \le i \le j \le g} f_{n+2i}f_{n+2j+1}\right)$$
$$+ \left(\frac{\ell}{2} - \sum_{i=1}^g \alpha_{n+2i}\right)f_n + \alpha_n\sum_{i=1}^g f_{n+2i}.$$

This is called the *higher order Painlevé equation of type* $A_{\ell-1}^{(1)}$; see [30].

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4.2 Affine Weyl group symmetry

Our goal here is to lift the Weyl group symmetry of the homogeneous τ-sequence (see Sect. 2.2) to the level of birational transformations of the Darboux chain (4.5).

First we prepare some relevant formulae, which will be used in Sect. 5 (P_{II} case) too. In a similar manner as Sect. 2.3, we can rewrite (2.19a) equivalently into

$$\tau_{i-1}(\boldsymbol{x} + \boldsymbol{u})\tau_{i+1}(\boldsymbol{x} - \boldsymbol{u}) = G_0(\hat{\tau}_i(\boldsymbol{x}), \tau_i(\boldsymbol{x}); \boldsymbol{u}), \tag{4.6}$$

where $\hat{\tau}_i = r_i(\tau_i)$. Regarding the right hand side, refer to (2.25). Let us consider the Taylor expansion of (4.6) in $\boldsymbol{u} = (u_1, u_2, \ldots)$. Taking the coefficients of $1(= \boldsymbol{u}^0)$, u_1, u_2, and $u_1{}^2$ thus yield the formulae

$$D_{x_1}\hat{\tau}_i \cdot \tau_i - \tau_{i-1}\tau_{i+1} = 0, \tag{4.7a}$$

$$D_{x_2}\hat{\tau}_i \cdot \tau_i + D_{x_1}\tau_{i-1} \cdot \tau_{i+1} = 0, \tag{4.7b}$$

$$\left(D_{x_1}{}^3 + 2D_{x_3}\right)\hat{\tau}_i \cdot \tau_i + 3D_{x_2}\tau_{i-1} \cdot \tau_{i+1} = 0, \tag{4.7c}$$

$$\left(D_{x_1}{}^3 - 4D_{x_3}\right)\hat{\tau}_i \cdot \tau_i + 3D_{x_1}{}^2\tau_{i-1} \cdot \tau_{i+1} = 0, \tag{4.7d}$$

respectively. Also we recall that $r_i(\tau_n) = \tau_n$ $(n \neq i)$.

Now, if we apply the reduction conditions under consideration, then we verify from (4.7a) and (4.7b) that

$$D_x\hat{\sigma}_i \cdot \sigma_i = \sigma_{i-1}\sigma_{i+1}, \tag{4.8}$$

$$\alpha_i\hat{\sigma}_i\sigma_i = (D_x + x)\sigma_{i-1} \cdot \sigma_{i+1} \tag{4.9}$$

with $\alpha_i = \hat{d}_i - d_i = d_{i-1} - 2d_i + d_{i+1} + 1$. It is easy to see from (4.4) that r_i should keep w_n $(n \neq i, i-1)$ invariant. The computation of $r_i(w_i)$ reads as

$$\begin{aligned}
r_i(w_i) &= \frac{x}{2} + \frac{D_x\hat{\sigma}_i \cdot \sigma_{i+1}}{\hat{\sigma}_i\sigma_{i+1}} = w_i + \frac{D_x\hat{\sigma}_i \cdot \sigma_i}{\hat{\sigma}_i\sigma_i} \\
&= w_i + \frac{\alpha_i\sigma_{i-1}\sigma_{i+1}}{(D_x + x)\sigma_{i-1} \cdot \sigma_{i+1}}, \quad \text{using (4.8) and (4.9)} \\
&= w_i + \frac{\alpha_i}{w_{i-1} + w_i}.
\end{aligned}$$

Similarly we find that $r_i(w_{i-1}) = w_{i-1} - \alpha_i/(w_{i-1} + w_i)$. Moreover, a cyclic permutation of the suffices $\pi : (\sigma_n, d_n) \mapsto (\sigma_{n+1}, d_{n+1})$ keeps the bilinear form (4.2) invariant and so does an inversion $\iota : (\sigma_n, d_n, x) \mapsto (\sigma_{-n}, d_{-n}, \sqrt{-1}x)$. It is again easy to lift π and ι to birational symmetries of (4.5). Summarizing above, we have the

Theorem 4.4. *The Darboux chain (4.5) is invariant under birational transformations r_i $(i \in \mathbb{Z}/\ell\mathbb{Z})$, π, and ι defined by*

	Action on α_n and x	Action on w_n
r_i	$\alpha_i \mapsto -\alpha_i,$	$w_{i-1} \mapsto w_{i-1} - \dfrac{\alpha_i}{w_{i-1} + w_i},$
	$\alpha_{i\pm1} \mapsto \alpha_{i\pm1} + \alpha_i$	$w_i \mapsto w_i + \dfrac{\alpha_i}{w_{i-1} + w_i}$
π	$\alpha_n \mapsto \alpha_{n+1}$	$w_n \mapsto w_{n+1}$
ι	$\alpha_n \mapsto \alpha_{-n}, x \mapsto \sqrt{-1}x$	$w_n \mapsto \sqrt{-1}w_{-n-1}$

Note that birational mappings $\langle r_i(i \in \mathbb{Z}/\ell\mathbb{Z})\rangle$ define a realization of affine Weyl group $W(A^{(1)}_{\ell-1})$. Additionally, π and ι realize respectively a rotation and an inversion of its Dynkin diagram.

4.3   Lax formalism

Introduce a function $\rho_n(\boldsymbol{x}, k) = X^+(k)\tau_n(\boldsymbol{x}) = \tau_n(\boldsymbol{x} - [k^{-1}])e^{\xi(\boldsymbol{x},k)}$. In order to construct linear differential equations satisfied by the wave function $\psi_n(\boldsymbol{x}, k) = \rho_n(\boldsymbol{x}, k)/\tau_n(\boldsymbol{x})$, let us first prepare some useful formulae. It follows from (2.20) multiplied by $1 \otimes X^+(k)$ that

$$k\tau_n \otimes \rho_{n+1} + \sum_{i+j=-1} X_i^- \tau_{n+1} \otimes X_j^+ \rho_n = 0. \tag{4.10}$$

Therefore, we find that (recall (2.25))

$$k\tau_n(\boldsymbol{x} + \boldsymbol{u})\rho_{n+1}(\boldsymbol{x} - \boldsymbol{u}, k) + G_0(\tau_{n+1}(\boldsymbol{x}), \rho_n(\boldsymbol{x}, k); \boldsymbol{u}) = 0.$$

The coefficients of $1(= \boldsymbol{u}^0)$, u_1, u_2, and u_1^2 in the above equation show respectively that

$$k\tau_n\rho_{n+1} + D_{x_1}\tau_{n+1} \cdot \rho_n = 0,$$
$$kD_{x_1}\tau_n \cdot \rho_{n+1} - D_{x_2}\tau_{n+1} \cdot \rho_n = 0,$$
$$3kD_{x_2}\tau_n \cdot \rho_{n+1} - \left(D_{x_1}^3 + 2D_{x_3}\right)\tau_{n+1} \cdot \rho_n = 0,$$
$$3kD_{x_1}^2\tau_n \cdot \rho_{n+1} - \left(D_{x_1}^3 - 4D_{x_3}\right)\tau_{n+1} \cdot \rho_n = 0.$$

Hence we have the following linear differential equations for ψ_n:

$$\frac{\partial \psi_n}{\partial x_1} = \frac{D_{x_1}\tau_{n+1} \cdot \tau_n}{\tau_{n+1}\tau_n}\psi_n + k\psi_{n+1}, \tag{4.11a}$$

$$\frac{\partial \psi_n}{\partial x_2} = \frac{D_{x_2}\tau_{n+1} \cdot \tau_n}{\tau_{n+1}\tau_n}\psi_n + k\frac{D_{x_1}\tau_{n+2} \cdot \tau_n}{\tau_{n+2}\tau_n}\psi_{n+1} + k^2\psi_{n+2}, \tag{4.11b}$$

$$\frac{\partial \psi_n}{\partial x_3} = \frac{1}{2}\frac{D_{x_3}\tau_{n+1} \cdot \tau_n}{\tau_{n+1}\tau_n}\psi_n + \frac{k}{2}\frac{\left(D_{x_1}^2 + D_{x_2}\right)\tau_{n+2} \cdot \tau_n}{\tau_{n+2}\tau_n}\psi_{n+1}$$
$$+ k^2\frac{D_{x_1}\tau_{n+3} \cdot \tau_n}{\tau_{n+3}\tau_n}\psi_{n+2} + k^3\psi_{n+3}. \tag{4.11c}$$

Note that it is possible to express $\partial\psi_n/\partial x_a$ as a linear sum of $\psi_n, \psi_{n+1}, \ldots, \psi_{n+a}$ for general $a \in \mathbb{Z}_{>0}$.

Now we require the homogeneity condition $E\tau_n = d_n\tau_n$ and set

$$\phi_n(x, k) = \psi_n((x, -1/2, 0, 0, \ldots), k).$$

In view of Lemma 3.11 and (4.3), we deduce from (4.11a) and (4.11b) the following

Lemma 4.5. *The wave functions $\phi_n = \phi_n(x, k)$ satisfy the following linear differential equations*:

$$k\frac{\partial}{\partial k}\phi_n = (d_{n+1} - d_n)\phi_n + k(w_n + w_{n+1})\phi_{n+1} - k^2\phi_{n+2}, \tag{4.12}$$

$$\frac{\partial}{\partial x}\phi_n = \left(\frac{x}{2} - w_n\right)\phi_n + k\phi_{n+1}. \tag{4.13}$$

The linear differential equation (4.12) with respect to the spectral variable k has a regular singularity at $k = 0$ and an irregular singularity (of Poincaré rank two) at $k = \infty$. While the latter (4.13) describes the monodromy preserving deformation of the former (4.12). We shall slightly modify this system. Let $\Phi = {}^{\mathrm{T}}\!\left(\phi_0, k\phi_1, \ldots, k^{\ell-1}\phi_{\ell-1}\right)$ and $z = k^\ell$. Then (4.12) is converted to the $\ell \times \ell$ matrix equation

$$\frac{\partial \Phi}{\partial z} = A\Phi = \left(C + \frac{A_0}{z}\right)\Phi, \tag{4.14}$$

where

$$C = \begin{pmatrix} & & O \\ -1 & & \\ h_0 & -1 & \end{pmatrix},$$

$$A_0 = \begin{pmatrix} e_0 & h_1 & -1 & & & \\ & e_1 & h_2 & \ddots & & \\ & & e_2 & \ddots & -1 & \\ & & & \ddots & h_{\ell-1} & \\ & & & & e_{\ell-1} \end{pmatrix}$$

with $e_n = (d_{n+1} - d_n + n)/\ell$ and $h_n = (w_n + w_{n-1})/\ell$. Note that the two singularities of (4.14) are $z = 0$ (regular singularity) and $z = \infty$ (irregular singularity of Poincaré rank $2/\ell$). On the other hand, the deformation equation (4.13) becomes

$$\frac{\partial \Phi}{\partial x} = B\Phi = \left(\begin{pmatrix} \frac{x}{2} - w_0 & 1 & & & \\ & \frac{x}{2} - w_1 & 1 & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ & & & & \frac{x}{2} - w_{\ell-1} \end{pmatrix} + z\begin{pmatrix} & & \\ & & \\ 1 & & \end{pmatrix}\right)\Phi. \tag{4.15}$$

We can recover the Darboux chain (4.5), as well as the case of P_{III}-chain (Sect. 3), from the integrability condition $\left[\frac{\partial}{\partial x} - B, \frac{\partial}{\partial z} - A\right] = 0$.

5 From KP hierarchy to Painlevé II equation

In this section we briefly review the derivation of the second Painlevé equation from the two-periodic mKP hierarchy, i.e., mKdV hierarchy, through a similarity reduction; cf. [1].

5.1 Similarity reduction

Let $\tau_n(\boldsymbol{x})$ be a solution of the mKP hierarchy, (2.12) or (2.21), satisfying the following conditions: (i) homogeneity $E\tau_n = d_n\tau_n$ ($d_n \in \mathbb{C}$); (ii) two-periodicity $\tau_{n+2} = \tau_n$; (iii) specialization $x_n = 0$ ($n \neq 1, 3$). We see that this specialization is meant to eliminate all time evolutions except the first nontrivial two. No generality is lost by taking

$$x_1 = x, \quad x_3 = -\frac{4}{3}, \quad \text{and} \quad x_n = 0 \quad (n \neq 1, 3). \tag{5.1}$$

Set $\sigma_n(x) = \tau_n(\boldsymbol{x})$.

Proposition 5.1. *A pair of functions* $\sigma_n = \sigma_n(x)$ $(n \in \mathbb{Z}/2\mathbb{Z})$ *satisfies*

$$D_x{}^2\sigma_n \cdot \sigma_{n+1} = 0, \tag{5.2}$$

$$\left(D_x{}^3 - xD_x + d_n - d_{n+1}\right)\sigma_n \cdot \sigma_{n+1} = 0. \tag{5.3}$$

Proof. Notice by (5.1) that

$$\frac{\partial}{\partial x_1} = \frac{\mathrm{d}}{\mathrm{d}x} \quad \text{and} \quad 4\frac{\partial}{\partial x_3} = x\frac{\mathrm{d}}{\mathrm{d}x} - E. \tag{5.4}$$

It is immediate to verify (5.2) and (5.3) from (2.27) and (2.28), respectively. □

Define functions $q = q(x)$ and $f_n = f_n(x)$ $(n \in \mathbb{Z}/2\mathbb{Z})$ as

$$q = \frac{D_x\sigma_0 \cdot \sigma_1}{\sigma_0\sigma_1}, \quad f_n = \frac{x}{2} - \frac{D_x{}^2\sigma_{n+1} \cdot \sigma_{n+1}}{\sigma_{n+1}{}^2}. \tag{5.5}$$

Let $\alpha_n = d_{n-1} - 2d_n + d_{n+1} + 1$. Due to Prop. 5.1 we then arrive at the

Theorem 5.2. *It holds that*

$$\frac{\mathrm{d}q}{\mathrm{d}x} = \frac{f_0 - f_1}{2}, \quad \frac{\mathrm{d}f_0}{\mathrm{d}x} = 2f_0q + \frac{\alpha_0}{2}, \quad \frac{\mathrm{d}f_1}{\mathrm{d}x} = -2f_1q + \frac{\alpha_1}{2}, \tag{5.6}$$

and $f_0 + f_1 - 2q^2 = x$.

Let $p = f_0$. Then (5.6) is converted to the Hamiltonian system

$$\frac{\mathrm{d}q}{\mathrm{d}x} = \frac{\partial H}{\partial p} \quad \text{and} \quad \frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{\partial H}{\partial q} \quad \text{with} \quad H = \frac{p^2}{2} - \left(q^2 + \frac{x}{2}\right)p - \frac{\alpha_0 q}{2},$$

which is equivalent to the second Painlevé equation P_{II}:

$$\frac{\mathrm{d}^2q}{\mathrm{d}x^2} = 2q^3 + xq + \frac{\alpha_0 - 1}{2}.$$

5.2 Affine Weyl group symmetry

First, it is clear that the permutation $\sigma_0 \leftrightarrow \sigma_1$ simultaneous with $d_0 \leftrightarrow d_1$ leaves the bilinear form (5.2) and (5.3) of P_{II} invariant. As a consequence we obtain a birational symmetry π : $(q, f_0, f_1, \alpha_0, \alpha_1) \mapsto (-q, f_1, f_0, \alpha_1, \alpha_0)$.

The next task is to realize the action of r_1, given in Sect 2.2, as a birational symmetry of P_{II}. Applying the reduction conditions to the formulae (4.7a), (4.7c), and (4.7d), we thus observe that

$$D_x\hat{\sigma}_1 \cdot \sigma_1 = \sigma_0{}^2, \tag{5.7}$$

$$\alpha_1\hat{\sigma}_1\sigma_1 = \left(-2D_x{}^2 + x\right)\sigma_0 \cdot \sigma_0. \tag{5.8}$$

Since f_1 and $f_0 + f_1 - 2q^2 (= x)$ are unchanged under r_1, it is enough to compute only $r_1(q)$. We have

$$r_1(q) = \frac{D_x \sigma_0 \cdot \hat{\sigma}_1}{\sigma_0 \hat{\sigma}_1} = q + \frac{D_x \sigma_1 \cdot \hat{\sigma}_1}{\sigma_1 \hat{\sigma}_1} = q - \frac{\sigma_0{}^2}{\sigma_1 \hat{\sigma}_1}, \quad \text{using (5.7)}$$

$$= q - \frac{\alpha_1 \sigma_0{}^2}{\left(-2D_x{}^2 + x\right)\sigma_0 \cdot \sigma_0}, \quad \text{using (5.8)}$$

$$= q - \frac{\alpha_1}{2f_1}.$$

The action of r_0 is obtained as a composition $\pi r_1 \pi^{-1}$. The result is summarized in the

Theorem 5.3. P_{II} *is invariant under birational transformations* r_i ($i = 0, 1$) *and* π *defined by*

	Action on α_n	Action on q	Action on f_n
r_0	$(\alpha_0, \alpha_1) \mapsto (-\alpha_0, \alpha_1 + 2\alpha_0)$	$q \mapsto q + \dfrac{\alpha_0}{2f_0}$	$f_1 \mapsto f_1 + \dfrac{2\alpha_0 q}{f_0} + \dfrac{\alpha_0{}^2}{2f_0{}^2}$
r_1	$(\alpha_0, \alpha_1) \mapsto (\alpha_0 + 2\alpha_1, -\alpha_1)$	$q \mapsto q - \dfrac{\alpha_1}{2f_1}$	$f_0 \mapsto f_0 - \dfrac{2\alpha_1 q}{f_1} + \dfrac{\alpha_1{}^2}{2f_1{}^2}$
π	$\alpha_0 \leftrightarrow \alpha_1$	$q \mapsto -q$	$f_0 \leftrightarrow f_1$

It can be verified straightforwardly that birational mappings $\langle r_0, r_1, \pi \rangle$ certainly fulfill the fundamental relations, $r_i{}^2 = \pi^2 = 1$ and $r_0 \pi = \pi r_1$, of affine Weyl group $W(A_1^{(1)})$, where π corresponds to the diagram automorphism.

5.3 Lax formalism

The homogeneity $E\tau_n = d_n \tau_n$ implies that $(E - k\partial/\partial k)\psi_n(x, k) = 0$; see Lemma 3.11. Let $\phi_n(x, k) = \psi_n((x, 0, -3/4, 0, 0, \ldots), k)$. By virtue of the two-periodicity $\tau_{n+2} = \tau_n$ and (5.4), the lemma below is immediate from (4.11a) and (4.11c).

Lemma 5.4. *The wave functions* $\phi_n = \phi_n(x, k)$ ($n \in \mathbb{Z}/2\mathbb{Z}$) *satisfy the following linear equations*:

$$k\frac{\partial}{\partial k}\phi_n = (d_{n+1} - d_n)\phi_n + 2kf_{n+1}\phi_{n+1} + 4(-1)^n k^2 q\phi_n - 4k^3 \phi_{n+1}, \tag{5.9}$$

$$\frac{\partial}{\partial x}\phi_n = (-1)^{n+1} q\phi_n + k\phi_{n+1}. \tag{5.10}$$

In fact, the above linear system (5.9) and (5.10) coincides with the Lax pair found by Flaschka and Newell [8]. We shall slightly modify it by taking $\Phi = {}^{\mathrm{T}}(\phi_0, k\phi_1)$ and $z = k^2$. Thus we have

$$\frac{\partial \Phi}{\partial z} = A\Phi = \left(\frac{1}{z}\begin{pmatrix} e_0 & f_1 \\ 0 & e_1 \end{pmatrix} + \begin{pmatrix} 2q & -2 \\ f_0 & -2q \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}\right)\Phi, \tag{5.11}$$

$$\frac{\partial \Phi}{\partial x} = B\Phi = \left(\begin{pmatrix} -q & 1 \\ 0 & q \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)\Phi, \tag{5.12}$$

where $e_0 = (d_1 - d_0)/2$ and $e_1 = (d_0 - d_1 + 1)/2$. The two singularities of (5.11) are $z = 0$ (regular singularity) and $z = \infty$ (irregular singularity of Poincaré rank 3/2). The latter (5.12) represents the monodromy preserving deformation of the former (5.11). Again one can derive P_{II}, (5.6), from the integrability condition $\left[\frac{\partial}{\partial x} - B, \frac{\partial}{\partial z} - A\right] = 0$ of the system.

6 UC hierarchy

In this section we review the UC hierarchy, which is an extension of the KP hierarchy proposed in [46]. We also present some functional equations arising from the UC hierarchy as preliminaries to the following two sections.

6.1 Universal character and UC hierarchy

For a pair of partitions λ and μ, the *universal character* $S_{[\lambda,\mu]} = S_{[\lambda,\mu]}(x, y)$ is a polynomial in $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ defined by the *twisted* Jacobi–Trudi formula (see [21]):

$$S_{[\lambda,\mu]}(x, y) = \det \begin{pmatrix} p_{\mu_{l'-i+1}+i-j}(y), & 1 \leq i \leq l' \\ p_{\lambda_{i-l'}-i+j}(x), & l' + 1 \leq i \leq l + l' \end{pmatrix}_{1 \leq i,j \leq l+l'} \tag{6.1}$$

with $l = l(\lambda)$ and $l' = l(\mu)$. If we count the degree of variables as $\deg x_n = n$ and $\deg y_n = -n$, then $S_{[\lambda,\mu]}$ is homogeneous of degree $|\lambda| - |\mu|$; i.e., $ES_{[\lambda,\mu]} = (|\lambda| - |\mu|)S_{[\lambda,\mu]}$. Here, we henceforth let E denote the Euler operator given as

$$E = \sum_{n=1}^{\infty} \left(nx_n \frac{\partial}{\partial x_n} - ny_n \frac{\partial}{\partial y_n} \right).$$

Note that the Schur function S_λ is a special case of the universal character: $S_\lambda(x) = \det(p_{\lambda_i-i+j}(x)) = S_{[\lambda,\emptyset]}(x, y)$.

Let us introduce the *vertex operators*

$$X^{\pm}(k) = \sum_{n \in \mathbb{Z}} X_n^{\pm} k^n = e^{\pm \xi(x - \widetilde{\partial}_y, k)} e^{\mp \xi(\widetilde{\partial}_x, k^{-1})},$$

$$Y^{\pm}(k^{-1}) = \sum_{n \in \mathbb{Z}} Y_n^{\pm} k^{-n} = e^{\pm \xi(y - \widetilde{\partial}_x, k^{-1})} e^{\mp \xi(\widetilde{\partial}_y, k)}.$$

The operators $X_i^{\pm}$ ($i \in \mathbb{Z}$) then satisfy the fermionic relations: $X_i^{\pm} X_j^{\pm} + X_{j-1}^{\pm} X_{i+1}^{\pm} = 0$ and $X_i^+ X_j^- + X_{j+1}^- X_{i-1}^+ = \delta_{i+j,0}$. The same relations hold also for $Y_i^{\pm}$, of course. Interestingly enough, $X_i^{\pm}$ and $Y_j^{\pm}$ mutually commute. By means of these operators, the universal character admits the following expression (cf. (2.4)):

$$S_{[\lambda,\mu]}(x, y) = X_{\lambda_1}^+ \ldots X_{\lambda_l}^+ Y_{\mu_1}^+ \ldots Y_{\mu_{l'}}^+.1. \tag{6.2}$$

Now we are ready to formulate the UC hierarchy.

Definition 6.1. For an unknown function $\tau = \tau(x, y)$, the system of bilinear relations

$$\sum_{i+j=-1} X_i^- \tau \otimes X_j^+ \tau = \sum_{i+j=-1} Y_i^- \tau \otimes Y_j^+ \tau = 0 \tag{6.3}$$

is called the *UC hierarchy*.

If $\tau = \tau(x, y)$ does not depend on $y = (y_1, y_2, \ldots)$, then the latter equality of (6.3) trivially holds and the former is reduced to the bilinear expression (2.5) of the KP hierarchy. From this aspect the UC hierarchy is literally an extension of the KP hierarchy. Moreover, as shown in [46] the totality

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of solutions of (6.3) forms a direct-product of two Sato Grassmannians and, in particular, the set of homogeneous polynomial solutions is equal to that of the universal characters.

It is obvious that (6.3) can be rewritten into the form

$$\frac{1}{2\pi\sqrt{-1}}\oint e^{\xi(x-x',z)}dz\, \tau(x' + [z^{-1}], y' + [z])\tau(x - [z^{-1}], y - [z]) = 0, \tag{6.4a}$$

$$\frac{1}{2\pi\sqrt{-1}}\oint e^{\xi(y-y',w)}dw\, \tau(x' + [w], y' + [w^{-1}])\tau(x - [w], y - [w^{-1}]) = 0 \tag{6.4b}$$

for arbitrary x, y, x', and y'. Let us try to write down a Hirota differential equation *naively* after the case of the KP hierarchy; cf. Sect. 2.3. Namely, consider the Taylor expansion of (6.4a) at $\{x' = x, y' = y\}$, i.e., replace (x', x, y', y) with $(x + u, x - u, y + v, y - v)$ and then expand with respect to $(u, v) = (u_1, u_2, \ldots, v_1, v_2, \ldots)$. Hence we obtain

$$\sum_{i+j+k=-1} p_i(-2u)p_{-j}(\tilde{\partial}_u)p_k(\tilde{\partial}_v)\tau(x + u, y + v)\tau(x - u, y - v) = 0.$$

Taking the coefficient of $1 = u^0 v^0$, for example, leads to

$$\sum_{i=0}^{\infty} p_{i+1}(\tilde{D}_x)p_i(\tilde{D}_y)\tau \cdot \tau = 0.$$

Unfortunately, every differential equation with respect to x and y contained in the UC hierarchy is of infinite order as well as the above one. This fact reflects that the integrand of (6.4a) with $x' = x$ and $y' = y$ may be singular not only at $z = 0$ but also at $z = \infty$, unlike the case of the KP hierarchy; cf. (2.6). But, however, it is possible to derive a closed functional equation from the UC hierarchy, (6.3) or (6.4), by a certain appropriate choice of parameters x, y, x', and y'; see Sect. 6.2 below.

It is also known that if $\tau = \tau(x, y)$ is a solution of (6.3), then so are $X^+(\alpha)\tau$ and $Y^+(\beta)\tau$ for arbitrary constants $\alpha, \beta \in \mathbb{C}^\times$. Now we are interested in the bilinear relations among the contiguous solutions connected by the vertex operators. For the UC hierarchy, a counterpart of the modified KP hierarchy (cf. Definition. 2.3) is introduced as follows:

Definition 6.2. Suppose $\tau_{m,n} = \tau_{m,n}(x, y)$ to be a solution of the UC hierarchy. Let

$$\tau_{m+1,n} = X^+(\alpha_m)\tau_{m,n}, \quad \tau_{m,n+1} = Y^+(\beta_n)\tau_{m,n}, \quad \tau_{m+1,n+1} = X^+(\alpha_m)Y^+(\beta_n)\tau_{m,n} = Y^+(\beta_n)X^+(\alpha_m)\tau_{m,n},$$

for arbitrary constants $\alpha_m, \beta_n \in \mathbb{C}^\times$. The whole set of functional equations satisfied by $\tau_{m,n}$'s are called the *modified UC hierarchy* or, shortly, *mUC hierarchy*.

Example 6.3. The mUC hierarchy includes the bilinear equations

$$\sum_{i+j=-2} X_i^- \tau_{m,n} \otimes X_j^+ \tau_{m+1,n} = \sum_{i+j=-1} Y_i^- \tau_{m,n} \otimes Y_j^+ \tau_{m+1,n} = 0, \tag{6.5}$$

$$\tau_{m,n} \otimes \tau_{m+1,n+1} - \sum_{i+j=0} X_i^- \tau_{m+1,n} \otimes X_j^+ \tau_{m,n+1} = \sum_{i+j=-2} Y_i^- \tau_{m+1,n} \otimes Y_j^+ \tau_{m,n+1} = 0. \tag{6.6}$$

Here the former and the latter can be deduced from (6.3) by applying $1 \otimes X^+(\alpha_m)$ and $X^+(\alpha_m) \otimes Y^+(\beta_n)$, respectively.

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Next we construct a sequence of homogeneous solutions of the UC hierarchy, which is crucial to reach for Painlevé equations. The argument will be proceeded along a parallel way with the case of the KP hierarchy; cf. Sect. 2.2. Let us first introduce partial differential operators $V_X(c)$ and $V_Y(c)$ equipped with a constant parameter $c \in \mathbb{C}$, defined by

$$V_X(c) = \int_\gamma X^+(k)k^{-c-1}\mathrm{d}k \quad \text{and} \quad V_Y(c) = \int_\gamma Y^+(k^{-1})k^{c-1}\mathrm{d}k,$$

where the path γ is again taken as well as Sect. 2.2. Let τ be a solution of the UC hierarchy, (6.3). Then one can verify that both $V_X(c)\tau$ and $V_Y(c)\tau$ solve (6.3); moreover, they satisfy the mUC hierarchy.

Lemma 6.4. *It holds that* $[E, V_X(c)] = cV_X(c)$ *and* $[E, V_Y(c)] = -cV_Y(c)$.

This lemma guarantees that V_X and V_Y preserve the homogeneity. Suppose $\tau_{0,0}$ to be a solution of the UC hierarchy (6.3) satisfying $E\tau_{0,0} = d_{0,0}\tau_{0,0}$. Define a sequence $\{\tau_{0,0}, \tau_{1,0}, \tau_{0,1}, \tau_{1,1}, \ldots\}$ recursively by $\tau_{m+1,n} = V_X(c_m)\tau_{m,n}$ and $\tau_{m,n+1} = V_Y(c'_n)\tau_{m,n}$ for arbitrary $c_m, c'_n \in \mathbb{C}$ given. Then $\tau_{m,n}$ becomes a solution of the mUC hierarchy still satisfying the homogeneity $E\tau_{m,n} = d_{m,n}\tau_{m,n}$, where $d_{m+1,n} = d_{m,n} + c_m$ and $d_{m,n+1} = d_{m,n} - c'_n$; thereby, the *balancing* condition

$$d_{m,n} + d_{m+1,n+1} = d_{m,n+1} + d_{m+1,n} \tag{6.7}$$

is fulfilled. Though we do not enter into details, we can find a mutually commuting pair of Weyl group actions of type A on this homogeneous τ-sequence; see [51].

Remark 6.5. Let us demonstrate one alternative to derive (6.7). Suppose $\tau_{m,n}$ to be a homogeneous solution of the mUC hierarchy. Therefore we have $\tau_{m,n} = c^{-d_{m,n}}\tau_{m,n}(cx_1, c^2 x_2, \ldots, c^{-1}y_1, c^{-2}y_2, \ldots)$ for some $d_{m,n} \in \mathbb{C}$ and arbitrary $c \in \mathbb{C}^\times$. Substituting this into (6.6), we thus obtain (6.7) as a necessary condition.

Example 6.6 (A sequence of universal characters). If we take $c = n$ to be an integer and γ a positively oriented small circle around $k = 0$, it follows that $V_X(n) = 2\pi\sqrt{-1}X_n^+$ and $V_Y(n) = 2\pi\sqrt{-1}Y_n^+$. Hence, starting from a trivial solution $\tau \equiv 1$ of the UC hierarchy, we can construct via (6.2) a homogeneous τ-sequence in terms of the universal characters. This type of polynomial solutions of the (modified) UC hierarchy gives rise to rational or algebraic solutions of Painlevé equations through the similarity reduction; cf. [47, 48].

6.2 Preliminaries for Sects. 7 and 8: difference/differential equations inside mUC hierarchy

Let δ_t and $\tilde\delta_t$ denote the vector fields

$$\delta_t = \sum_{n=1}^\infty \left(t^n \frac{\partial}{\partial x_n} - t^{-n}\frac{\partial}{\partial y_n} \right) \quad \text{and} \quad \tilde\delta_t = \sum_{n=1}^\infty \left(nt^n \frac{\partial}{\partial x_n} + nt^{-n}\frac{\partial}{\partial y_n} \right) \tag{6.8}$$

hereafter. For a vector field v, let D_v stand for its corresponding Hirota differential. We first summarize some difference and/or differential equations arising from the mUC hierarchy.

Lemma 6.7. *The following functional equations hold*:

$$(t - s)\tau_{m,n}(\boldsymbol{x} - [t] - [s], \boldsymbol{y} - [t^{-1}] - [s^{-1}])\tau_{m+1,n+1}(\boldsymbol{x}, \boldsymbol{y})$$
$$- t\tau_{m,n+1}(\boldsymbol{x} - [t], \boldsymbol{y} - [t^{-1}])\tau_{m+1,n}(\boldsymbol{x} - [s], \boldsymbol{y} - [s^{-1}])$$
$$+ s\tau_{m,n+1}(\boldsymbol{x} - [s], \boldsymbol{y} - [s^{-1}])\tau_{m+1,n}(\boldsymbol{x} - [t], \boldsymbol{y} - [t^{-1}]) = 0, \tag{6.9}$$

$$(D_{\delta_t} - 1)\,\tau_{m,n+1}(\boldsymbol{x}, \boldsymbol{y}) \cdot \tau_{m+1,n}(\boldsymbol{x}, \boldsymbol{y})$$
$$+ \tau_{m,n}(\boldsymbol{x} - [t], \boldsymbol{y} - [t^{-1}])\tau_{m+1,n+1}(\boldsymbol{x} + [t], \boldsymbol{y} + [t^{-1}]) = 0, \tag{6.10}$$

$$\left(D_{\delta_t} + \frac{t}{s - t}\right)\tau_{m,n}(\boldsymbol{x} - [s], \boldsymbol{y} - [s^{-1}]) \cdot \tau_{m+1,n}(\boldsymbol{x}, \boldsymbol{y})$$
$$+ \frac{t}{t - s}\tau_{m,n}(\boldsymbol{x} - [t], \boldsymbol{y} - [t^{-1}])\tau_{m+1,n}(\boldsymbol{x} + [t] - [s], \boldsymbol{y} + [t^{-1}] - [s^{-1}]) = 0. \tag{6.11}$$

We refer the reader to [51] where a more general class of functional relations of the UC hierarchy including the above is established. See also [49, Appendix] regarding (6.9). We mention that the first two equations and the last are originated from (6.6) and (6.5), respectively. Furthermore, if we differentiate (6.10) with respect to t, then we get

$$D_{\tilde{\delta}_t}\tau_{m,n+1}(\boldsymbol{x}, \boldsymbol{y}) \cdot \tau_{m+1,n}(\boldsymbol{x}, \boldsymbol{y})$$
$$- D_{\delta_t}\tau_{m,n}(\boldsymbol{x} - [t], \boldsymbol{y} - [t^{-1}]) \cdot \tau_{m+1,n+1}(\boldsymbol{x} + [t], \boldsymbol{y} + [t^{-1}]) = 0. \tag{6.12}$$

Such bilinear relations as (6.9)–(6.12) generate the auxiliary system of linear equations associated with the UC hierarchy. Let us introduce the *wave function* $\psi_{m,n} = \psi_{m,n}(\boldsymbol{x}, \boldsymbol{y}, k)$ defined by

$$\psi_{m,n}(\boldsymbol{x}, \boldsymbol{y}, k) = \frac{\tau_{m,n-1}(\boldsymbol{x} - [k^{-1}], \boldsymbol{y} - [k])}{\tau_{m,n}(\boldsymbol{x}, \boldsymbol{y})}e^{\xi(\boldsymbol{x},k)}. \tag{6.13}$$

Note that if $\tau_{m,n}$ does not depend on variables $\boldsymbol{y} = (y_1, y_2, \ldots)$ and n then $\psi_{m,n}$ reduces to the wave function of the KP hierarchy; cf. (3.22). We list below a few of the linear equations that are relevant to us.

Lemma 6.8. *The following linear functional equations hold*:

$$\psi_{m,n} = \frac{\overline{\tau}_{m,n+1}\tau_{m+1,n-1}}{\overline{\tau}_{m+1,n}\tau_{m,n}}\overline{\psi}_{m,n+1} - tk\overline{\psi}_{m+1,n}, \tag{6.14}$$

$$\delta_t\psi_{m,n} + \frac{D_{\delta_t}\tau_{m,n} \cdot \tau_{m+1,n-1}}{\tau_{m,n}\tau_{m+1,n-1}}\psi_{m,n} - tk\frac{\tau_{m,n-1}\overline{\tau}_{m+1,n}}{\tau_{m,n}\tau_{m+1,n-1}}\overline{\psi}_{m+1,n} = 0, \tag{6.15}$$

$$\tilde{\delta}_t\psi_{m,n} + \frac{D_{\tilde{\delta}_t}\tau_{m,n} \cdot \tau_{m+1,n-1}}{\tau_{m,n}\tau_{m+1,n-1}}\psi_{m,n} + tk\frac{(D_{\delta_t} - 1)\,\underline{\tau}_{m,n-1} \cdot \overline{\tau}_{m+1,n}}{\tau_{m,n}\tau_{m+1,n-1}}\overline{\psi}_{m+1,n}$$
$$- tk\frac{\underline{\tau}_{m,n-1}\overline{\tau}_{m+1,n}}{\tau_{m,n}\tau_{m+1,n-1}}\delta_t\overline{\psi}_{m+1,n} = 0. \tag{6.16}$$

Here, for a function $f = f(\boldsymbol{x}, \boldsymbol{y})$ we abbreviate $f(\boldsymbol{x} + [t], \boldsymbol{y} + [t^{-1}])$ and $f(\boldsymbol{x} - [t], \boldsymbol{y} - [t^{-1}])$ to $\overline{f}$ and $\underline{f}$, respectively.

Proof. If we put $s = 1/k$ in (6.9) and (6.11), then we verify (6.14) and (6.15) immediately. Moreover, differentiating (6.15) with respect to t yields (6.16). $\qquad\square$

7 From UC hierarchy to Painlevé V chain

This section concerns a similarity reduction of the periodic mUC hierarchy. As a result we obtain the higher order Painlevé equation $P(A^{(1)}_{2\ell-1})$ or, equivalently, the Darboux chain with period 2ℓ ($\ell \geq 2$); cf. Sect. 4.

7.1 Similarity reduction

Let $\tau_{m,n} = \tau_{m,n}(x, y)$ be a solution of the mUC hierarchy. Assume the (ℓ_1, ℓ_2)-periodicity $\tau_{m+\ell_1,n} = \tau_{m,n+\ell_2} = \tau_{m,n}$ and the homogeneity

$$E\tau_{m,n}(x, y) = d_{m,n}\tau_{m,n}(x, y) \quad (d_{m,n} \in \mathbb{C}) \tag{7.1}$$

where $E = \sum_{n=1}^{\infty}(nx_n\partial/\partial x_n - ny_n\partial/\partial y_n)$. Note that the constants $d_{m,n}$ necessarily satisfy the balancing condition $d_{m,n} + d_{m+1,n+1} = d_{m,n+1} + d_{m+1,n}$; see Remark 6.5. As seen later we can in fact restrict ourselves to the case where $\ell_1 = \ell_2$, without loss of generality; nonetheless we shall consider a general case for a while. Now, we introduce the functions $\sigma_{m,n} = \sigma_{m,n}(a, s)$ ($m, n \in \mathbb{Z}/\ell\mathbb{Z}$) defined by $\sigma_{m,n}(a, s) = \tau_{m,n}(x, y)$ under the substitution

$$x_n = s + \frac{a}{n} \quad \text{and} \quad y_n = -s + \frac{a}{n}. \tag{7.2}$$

Note that s will play a role of the independent variable. From now on, we use abbreviated notations $\overline{\sigma}_{m,n}$ and $\underline{\sigma}_{m,n}$ which stand respectively for $\sigma_{m,n}(a + 1, s)$ and $\sigma_{m,n}(a - 1, s)$, while $\sigma_{m,n} = \sigma_{m,n}(a, s)$.

Proposition 7.1. *The functions $\sigma_{m,n} = \sigma_{m,n}(a, s)$ satisfy the system of bilinear equations*

$$(D_s + 1)\,\sigma_{m,n-1} \cdot \sigma_{m-1,n} = \underline{\sigma}_{m-1,n-1}\overline{\sigma}_{m,n}, \tag{7.3}$$

$$sD_s\underline{\sigma}_{m-1,n-1} \cdot \overline{\sigma}_{m,n} = (aD_s + d_{m-1,n} - d_{m,n-1})\,\sigma_{m,n-1} \cdot \sigma_{m-1,n}. \tag{7.4}$$

Proof. We observe that

$$\frac{\mathrm{d}}{\mathrm{d}s} = \sum_{n=1}^{\infty}\left(\frac{\mathrm{d}x_n}{\mathrm{d}s}\frac{\partial}{\partial x_n} + \frac{\mathrm{d}y_n}{\mathrm{d}s}\frac{\partial}{\partial y_n}\right) = \sum_{n=1}^{\infty}\left(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial y_n}\right) = \delta_1, \tag{7.5}$$

and also that

$$E = \sum_{n=1}^{\infty}\left(nx_n\frac{\partial}{\partial x_n} - ny_n\frac{\partial}{\partial y_n}\right) = \sum_{n=1}^{\infty}\left((ns + a)\frac{\partial}{\partial x_n} + (ns - a)\frac{\partial}{\partial y_n}\right) = s\tilde{\delta}_1 + a\delta_1. \tag{7.6}$$

Via the homogeneity (7.1), we rapidly verify (7.3) and (7.4) from (6.10) and (6.12), respectively. $\square$

The next task is to derive a system of nonlinear ordinary differential equations from the bilinear equations given in Prop. 7.1. We introduce a set of dependent variables $f_{m,n} = f_{m,n}(a, s)$ and $g_{m,n} = g_{m,n}(a, s)$ defined by

$$f_{m,n} = \frac{\overline{\sigma}_{m,n-1}\sigma_{m-1,n-1}}{\overline{\sigma}_{m-1,n}\sigma_{m,n-2}}, \quad g_{m,n} = \frac{\underline{\sigma}_{m-1,n-1}\overline{\sigma}_{m,n}}{\sigma_{m-1,n}\sigma_{m,n-1}}. \tag{7.7}$$

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Let ℓ denote the least common multiple of ℓ_1 and ℓ_2. One can then find the conservation laws:

$$\prod_{i=1}^{\ell} f_{m+i,n-i} = 1 \quad \text{and} \quad \sum_{i=1}^{\ell} g_{m+i,n-i} = \ell, \tag{7.8}$$

where the former is trivial and the latter a consequence of (7.3). Additionally, we consider another set of variables $u_{m,n} = u_{m,n}(a, s)$ and $v_{m,n} = v_{m,n}(a, s)$ defined by

$$u_{m,n} = \frac{D_s \sigma_{m,n} \cdot \overline{\sigma}_{m+1,n}}{\sigma_{m,n} \overline{\sigma}_{m+1,n}}, \quad v_{m,n} = -\frac{D_s \sigma_{m,n} \cdot \overline{\sigma}_{m,n+1}}{\sigma_{m,n} \overline{\sigma}_{m,n+1}}. \tag{7.9}$$

Note that $u_{m,n}$ and $v_{m,n}$ are identical each other if we interchange the roles of suffixes m and n and replace s with $-s$.

We know from (7.3) that

$$g_{m,n} = 1 - u_{m-1,n} - v_{m,n-1} \quad \text{and} \quad \overline{g}_{m,n} = 1 - u_{m-1,n-1} - v_{m-1,n-1}. \tag{7.10}$$

Although the above linear equations themselves can not be solved for $u_{m,n}$ and $v_{m,n}$, one can express $u_{m,n}$ and $v_{m,n}$ in terms of $g_{m,n}$ and $\overline{g}_{m,n}$ conversely. To be specific, we state the

Lemma 7.2. *Variables $u_{m,n}$ and $v_{m,n}$ are quadratic polynomials in $g_{m,n}$ and $\overline{g}_{m,n}$.*

Proof. First, the logarithmic derivative of $g_{m,n}$ reads

$$\frac{1}{g_{m,n}} \frac{\mathrm{d}g_{m,n}}{\mathrm{d}s} = -\frac{D_s \sigma_{m,n-1} \cdot \overline{\sigma}_{m,n}}{\sigma_{m,n-1} \overline{\sigma}_{m,n}} + \frac{D_s \underline{\sigma}_{m-1,n-1} \cdot \sigma_{m-1,n}}{\underline{\sigma}_{m-1,n-1} \sigma_{m-1,n}}$$

$$= v_{m,n-1} - \underline{v}_{m-1,n-1}. \tag{7.11}$$

Observe that

$$\begin{aligned}
u_{m-1,n} - \underline{v}_{m-1,n-1} &= \frac{D_s \underline{\sigma}_{m-1,n-1} \cdot \overline{\sigma}_{m,n}}{\underline{\sigma}_{m-1,n-1} \overline{\sigma}_{m,n}} \\
&= \frac{(aD_s + d_{m-1,n} - d_{m,n-1}) \sigma_{m,n-1} \cdot \sigma_{m-1,n}}{s \underline{\sigma}_{m-1,n-1} \overline{\sigma}_{m,n}}, \quad \text{using (7.4)} \\
&= \frac{a(g_{m,n} - 1) + d_{m-1,n} - d_{m,n-1}}{s g_{m,n}}, \quad \text{using (7.3) and (7.7).} \tag{7.12}
\end{aligned}$$

Thereby, with (7.10) in mind, each $u_{m,n}(a \pm 1)$ and $v_{m,n}(a \pm 1)$ can be written as a rational function in $u_{m,n}(a)$ and $v_{m,n}(a)$. Now, (7.11) combined with (7.12) yields

$$\begin{aligned}
\frac{\mathrm{d}g_{m,n}}{\mathrm{d}s} &= (v_{m,n-1} - u_{m-1,n})g_{m,n} + \frac{a(g_{m,n} - 1) + d_{m-1,n} - d_{m,n-1}}{s} \\
&= (2v_{m,n-1} + g_{m,n} - 1)g_{m,n} + \frac{a(g_{m,n} - 1) + d_{m-1,n} - d_{m,n-1}}{s}, \quad \text{using (7.10).} \tag{7.13}
\end{aligned}$$

Due to the conservation (7.8) and the periodicity, we hence obtain the equality

$$0 = \sum_{j=1}^{\ell} \frac{\mathrm{d}g_{m+j,n-j}}{\mathrm{d}s} = \sum_{j=1}^{\ell} (2v_{m+j,n-j-1} + g_{m+j,n-j} - 1)g_{m+j,n-j}. \tag{7.14}$$

On the other hand, it follows again from (7.10) that

$$v_{m+j,n-j-1} = v_{m,n-1} + \sum_{i=1}^{j} \left(\overline{g}_{m+i,n-i+1} - g_{m+i,n-i} \right)$$

for $1 \le j \le \ell$. Substituting this into (7.14) we obtain a linear equation for $v_{m,n-1}$; thus, we have

$$v_{m,n-1} = \frac{\ell+1}{2} - \frac{1}{\ell} \sum_{1 \le i \le j \le \ell} \overline{g}_{m+i,n-i+1} g_{m+j,n-j}.$$

By (7.10), we can modify this into the form

$$u_{m,n} = \frac{1-\ell}{2} + \frac{1}{\ell} \sum_{1 \le i \le j \le \ell-1} \overline{g}_{m+i+1,n-i+1} g_{m+j+1,n-j}. \tag{7.15}$$

Interchanging the roles of suffixes m and n in the above formula, we see also that

$$v_{m,n} = \frac{1-\ell}{2} + \frac{1}{\ell} \sum_{1 \le i \le j \le \ell-1} \overline{g}_{m-i+1,n+i+1} g_{m-j,n+j+1}$$

$$= \frac{1-\ell}{2} + \frac{1}{\ell} \sum_{1 \le i \le j \le \ell-1} g_{m+i,n-i+1} \overline{g}_{m+j+1,n-j+1}, \quad \text{using the periodicity.} \tag{7.16}$$

The lemma is proven. □

In fact, one can adopt $(u_{m,n}, v_{m,n})$ or, alternatively, $(g_{m,n}, \overline{g}_{m,n})$ as essential dependent variables because: if once $(u_{m,n}, v_{m,n})$ or $(g_{m,n}, \overline{g}_{m,n})$ are given then $f_{m,n}$ is determined from the first order linear equation

$$\frac{1}{f_{m,n}} \frac{\mathrm{d}f_{m,n}}{\mathrm{d}s} = \frac{D_s \sigma_{m-1,n-1} \cdot \overline{\sigma}_{m-1,n}}{\sigma_{m-1,n-1} \overline{\sigma}_{m-1,n}} - \frac{D_s \sigma_{m,n-2} \cdot \overline{\sigma}_{m,n-1}}{\sigma_{m,n-2} \overline{\sigma}_{m,n-1}}$$

$$= v_{m,n-2} - v_{m-1,n-1}$$

$$= \overline{g}_{m,n} - g_{m,n-1}$$

by quadrature.

Starting from the logarithmic derivative of $\overline{g}_{m,n}$:

$$\frac{1}{\overline{g}_{m,n}} \frac{\mathrm{d}\overline{g}_{m,n}}{\mathrm{d}s} = u_{m-1,n-1} - \overline{u}_{m-1,n},$$

we arrive at

$$\frac{\mathrm{d}\overline{g}_{m,n}}{\mathrm{d}s} = (u_{m-1,n-1} - v_{m-1,n-1}) \overline{g}_{m,n} - \frac{(a+1)(\overline{g}_{m,n}-1) + d_{m-1,n} - d_{m,n-1}}{s} \tag{7.17}$$

via (7.12) in the same manner as before; cf. (7.13). Now the necessary differential equations have been all present. Since (7.15) and (7.16) hold, one can eliminate $u_{m,n}$ and $v_{m,n}$ from (7.13) and (7.17) and thus obtain the differential equations for unknowns $(g_{m,n}, \overline{g}_{m,n})$. Alternatively, eliminating $g_{m,n}$

and $\bar{g}_{m,n}$ from (7.13) and (7.17) by (7.10) yields the system of differential equations for $(u_{m,n}, v_{m,n})$:

$$\frac{d}{ds}(u_{m-1,n+1} + v_{m,n}) = v_{m,n}^2 - u_{m-1,n+1}^2 - \left(1 - \frac{a}{s}\right)v_{m,n} + \left(1 + \frac{a}{s}\right)u_{m-1,n+1}$$

$$+ \frac{d_{m,n} - d_{m-1,n+1}}{s}, \tag{7.18a}$$

$$\frac{d}{ds}(v_{m,n} + u_{m,n}) = u_{m,n}^2 - v_{m,n}^2 - \left(1 + \frac{a+1}{s}\right)u_{m,n} + \left(1 - \frac{a+1}{s}\right)v_{m,n}$$

$$+ \frac{d_{m,n+1} - d_{m+1,n}}{s}. \tag{7.18b}$$

Interestingly enough, for each (m, n) given (7.18) is closed with respect to the 2ℓ-tuple of variables $(u_{m+i,n-i}, v_{m+i,n-i})$ where $i \in \mathbb{Z}/\ell\mathbb{Z}$. For instance, we fix $m = n = 0$ hereafter. To improve the representation we consider further change of variables

$$x = -\sqrt{-2s}, \quad w_{2i} = \sqrt{-2s}\,u_{i,-i} + \frac{s + a + \frac{1}{2}}{\sqrt{-2s}}, \quad w_{2i-1} = \sqrt{-2s}\,v_{i,-i} + \frac{s - a - \frac{1}{2}}{\sqrt{-2s}}. \tag{7.19}$$

Then we are led to the

Theorem 7.3. *The 2ℓ-tuple of functions $w_n = w_n(x)$ ($n \in \mathbb{Z}/2\ell\mathbb{Z}$) satisfies*

$$\frac{d}{dx}(w_n + w_{n-1}) = w_n^2 - w_{n-1}^2 + \alpha_n, \tag{7.20}$$

where $\alpha_{2i} = 2(d_{i+1,-i} - d_{i,-i+1} + a + 1)$ and $\alpha_{2i-1} = 2(d_{i-1,-i+1} - d_{i,-i} - a)$.

This is exactly the Darboux chain with period 2ℓ (see Sect. 4) and thus it is equivalent to the fifth Painlevé equation P_V when $\ell = 2$.

Remark 7.4. No generality is lost by assuming $\ell_1 = \ell_2$ because the system with general (ℓ_1, ℓ_2)-periodicity is obviously a special case of that with (ℓ, ℓ)-periodicity, provided ℓ is the least common multiple of ℓ_1 and ℓ_2.

7.2 Lax formalism

In view of Remark 7.4, we let $\ell_1 = \ell_2 = \ell$ from now on. Our goal here is to derive an auxiliary linear problem for $P(A_{2\ell-1}^{(1)})$ from the UC hierarchy via the similarity reduction. Recall the definition (6.13) of the wave function:

$$\psi_{m,n}(\boldsymbol{x}, \boldsymbol{y}, k) = \frac{\tau_{m,n-1}(\boldsymbol{x} - [k^{-1}], \boldsymbol{y} - [k])}{\tau_{m,n}(\boldsymbol{x}, \boldsymbol{y})} e^{\xi(\boldsymbol{x},k)}.$$

As analogous to Lemma 3.11, we have the

Lemma 7.5. *If $\tau_{m,n} = \tau_{m,n}(\boldsymbol{x}, \boldsymbol{y})$ obeys the homogeneity $E\tau_{m,n} = d_{m,n}\tau_{m,n}$. Then it holds that*

$$\left(E - k\frac{\partial}{\partial k}\right)\psi_{m,n} = (d_{m,n-1} - d_{m,n})\psi_{m,n}. \tag{7.21}$$

Set $\phi_{m,n}(a, s, k) = \psi_{m,n}(x, y, k)$ under the substitution (7.2).

Lemma 7.6. *The wave functions* $\phi_{m,n} = \phi_{m,n}(a) = \phi_{m,n}(a, s, k)$ *satisfy the following linear equations*:

$$\phi_{m,n}(a) = \frac{1}{f_{m+1,n+1}}\phi_{m,n+1}(a + 1) - k\phi_{m+1,n}(a + 1), \tag{7.22}$$

$$\frac{\partial}{\partial s}\phi_{m,n}(a) = (g_{m+1,n} - 1)\phi_{m,n}(a) + kg_{m+1,n}\phi_{m+1,n}(a + 1), \tag{7.23}$$

$$k\frac{\partial}{\partial k}\phi_{m,n}(a) = (d_{m+1,n-1} - d_{m,n-1})\phi_{m,n}(a)$$
$$+ k\left(sg_{m+1,n} + a + d_{m+1,n-1} - d_{m,n}\right)\phi_{m+1,n}(a + 1)$$
$$+ ksg_{m+1,n}\frac{\partial}{\partial s}\phi_{m+1,n}(a + 1). \tag{7.24}$$

Proof. It is immediate to deduce (7.22) and (7.23) respectively from (6.14) and (6.15) with $t = 1$ through (7.5). We can verify (7.24) from (6.16) by taking into account (7.6) and Lemma 7.5. □

Notice the equality

$$f_{m,n} = \frac{\overline{f}_{m+1,n}\overline{g}_{m,n}}{\overline{g}_{m+1,n-1}}, \tag{7.25}$$

which follows clearly from (7.7). The periodicity $\phi_{m+\ell,n} = \phi_{m,n+\ell} = \phi_{m,n}$ allows us to solve the linear equations (7.22) for $\overline{\phi}_{m,n} = \phi_{m,n}(a + 1)$; we thus find that

$$\overline{\phi}_{m,n} = \frac{1}{1 - k^\ell}\sum_{j=1}^{\ell} k^{j-1}\left(\prod_{i=1}^{j} f_{m+i,n-i+1}\right)\phi_{m+j-1,n-j}. \tag{7.26}$$

Moreover, applying the above formula twice shows that

$$\overline{\overline{\phi}}_{m,n} = \phi_{m,n}(a + 2)$$
$$= \frac{1}{(1 - k^\ell)^2}\sum_{j=1}^{\ell}\sum_{j'=1}^{\ell} k^{j+j'-2}\left(\prod_{i=1}^{j} \overline{f}_{m+i,n-i+1}\right)\left(\prod_{i'=1}^{j'} f_{m+j+i'-1,n-j-i'+1}\right)\phi_{m+j+j'-2,n-j-j'}$$
$$= \frac{f_{m,n}}{(1 - k^\ell)^2\overline{g}_{m,n}}\sum_{j=1}^{\ell}\sum_{j'=1}^{\ell} k^{j+j'-2}\overline{g}_{m+j,n-j}\left(\prod_{i=1}^{j+j'-1} f_{m+i,n-i}\right)\phi_{m+j+j'-2,n-j-j'}, \quad \text{using (7.25)}$$
$$= \frac{f_{m,n}}{(1 - k^\ell)^2\overline{g}_{m,n}}\sum_{r=1}^{2\ell-1} k^{r-1}C_r\phi_{m+r-1,n-r-1}, \quad \text{taking } r = j + j' - 1, \tag{7.27}$$

with

$$C_r = \begin{cases} \sum_{j=1}^{r}\overline{g}_{m+j,n-j}\prod_{i=1}^{r} f_{m+i,n-i} & (\text{if } 1 \le r \le \ell) \\ \sum_{j=r-\ell+1}^{\ell}\overline{g}_{m+j,n-j}\prod_{i=1}^{r} f_{m+i,n-i} & (\text{if } \ell \le r \le 2\ell - 1). \end{cases}$$

Here we have used the conservation (7.8).

Firstly, we shall take interest in a differential equation with respect to the spectral variable k. With the aid of (7.22), (7.23), and (7.25), one can rewrite (7.24) into

$$k\frac{\partial}{\partial k}\phi_{m,n} = (d_{m+1,n-1} - d_{m,n-1})\phi_{m,n} + k(a + d_{m+1,n-1} - d_{m,n})\overline{\phi}_{m+1,n}$$

$$+ ks\frac{g_{m+1,n}\overline{g}_{m+1,n+1}}{f_{m+1,n+1}}\overline{\overline{\phi}}_{m+1,n+1}.$$

Hence, eliminating $\overline{\phi}_{m+1,n}$ and $\overline{\overline{\phi}}_{m+1,n+1}$ by (7.26) and (7.27) yields the equation

$$k\frac{\partial}{\partial k}\phi_{m,n} = (d_{m+1,n-1} - d_{m,n-1})\phi_{m,n}$$

$$+ \frac{a + d_{m+1,n-1} - d_{m,n}}{1 - k^{\ell}}\sum_{j=1}^{\ell}k^{j}\left(\prod_{i=1}^{j}f_{m+i+1,n-i+1}\right)\phi_{m+j,n-j}$$

$$+ \frac{sg_{m+1,n}}{(1-k^{\ell})^2}\sum_{j=1}^{\ell}k^{j}\sum_{i'=1}^{j}\overline{g}_{m+i'+1,n-i'+1}\left(\prod_{i=1}^{j}f_{m+i+1,n-i+1}\right)\phi_{m+j,n-j}$$

$$+ \frac{sg_{m+1,n}}{(1-k^{\ell})^2}\sum_{j=\ell+1}^{2\ell-1}k^{j}\sum_{i'=j-\ell+1}^{\ell}\overline{g}_{m+i'+1,n-i'+1}\left(\prod_{i=1}^{j}f_{m+i+1,n-i+1}\right)\phi_{m+j,n-j}. \qquad (7.28)$$

This has two regular singularities at $k = 0, \infty$ and ℓ irregular singularities at $k = \exp\left(2\pi\sqrt{-1}n/\ell\right)$ ($n \in \mathbb{Z}/\ell\mathbb{Z}$). Notice that for each (m, n) fixed (7.28) is closed with respect to $\phi_{m+i,n-i}$ ($i \in \mathbb{Z}/\ell\mathbb{Z}$) as before. Let us consider

$$\Phi = {}^{\mathrm{T}}\left(\phi_{0,0}, k\phi_{1,-1}, k^2\phi_{2,-2}, \ldots, k^{\ell-1}\phi_{\ell-1,-\ell+1}\right) \quad \text{and} \quad z = k^{\ell}.$$

Then (7.28) takes the form

$$\frac{\partial\Phi}{\partial z} = A\Phi = \left(\frac{A_0}{z} + \frac{A_{1,0}}{z-1} + \frac{A_{1,-1}}{(z-1)^2}\right)\Phi, \qquad (7.29)$$

where the $\ell \times \ell$ matrices A_0, $A_{1,0}$, and $A_{1,-1}$ are given by

$$A_0 = \begin{pmatrix} e_0 & \zeta_{0,1} + \eta_{0,1} & \cdots & \zeta_{0,\ell-1} + \eta_{0,\ell-1} \\ & e_1 & \ddots & \vdots \\ & & \ddots & \zeta_{\ell-2,\ell-1} + \eta_{\ell-2,\ell-1} \\ & & & e_{\ell-1} \end{pmatrix},$$

$$A_{1,0} = -\left(\zeta_{i,j} + \eta_{i,j}\right)_{0\le i,j\le\ell-1} + \begin{pmatrix} \xi_{0,0} & & & O \\ \xi_{1,0} & \xi_{1,1} & & \\ \vdots & \ddots & \ddots & \\ \xi_{\ell-1,0} & \xi_{\ell-1,1} & \cdots & \xi_{\ell-1,\ell-1} \end{pmatrix},$$

$$A_{1,-1} = \left(\xi_{i,j}\right)_{0\le i,j\le\ell-1}$$

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with

$$e_n = \frac{d_{n+1,-n-1} - d_{n,-n-1} + n}{\ell},$$

$$\zeta_{n,n+j} = \frac{a + d_{n+1,-n-1} - d_{n,-n}}{\ell} \prod_{i=1}^{j} f_{n+i+1,-n-i+1},$$

$$\eta_{n,n+j} = \frac{s g_{n+1,-n}}{\ell} \sum_{i'=1}^{j} \bar{g}_{n+i'+1,-n-i'+1} \prod_{i=1}^{j} f_{n+i+1,-n-i+1},$$

$$\xi_{n,n+j} = s g_{n+1,-n} \prod_{i=1}^{j} f_{n+i+1,-n-i+1}$$

for $0 \le n \le \ell - 1$ and $1 \le j \le \ell$. Note that the suffix of each variable should be regarded as an element of $\mathbb{Z}/\ell\mathbb{Z}$. We observe that (7.29) has an irregular singularity (of Poincaré rank one) at $z = 1$, and has two regular singularities $z = 0$ and $z = \infty$ whose exponents read respectively $(e_0, e_1, \ldots, e_{\ell-1})$ and $(\rho_0, \rho_1, \ldots, \rho_{\ell-1})$ with $\rho_n = (a + d_{n,-n-1} - d_{n,-n} - n)/\ell$. From a standpoint of monodromy preserving deformations, (7.29) is the original linear differential equation that will be deformed with keeping its monodromy matrices and Stokes multipliers invariant.

Secondly, we derive from (7.23) the deformation equation

$$\frac{\partial}{\partial s}\Phi = B\Phi, \tag{7.30}$$

where

$$B = \text{diag}\,(g_{i+1,-i} - 1)_{0 \le i \le \ell - 1}$$

$$+ \frac{1}{s(1-z)}\begin{pmatrix} 0 & \xi_{0,1} & \cdots & \xi_{0,\ell-1} \\ & 0 & \ddots & \vdots \\ & & \ddots & \xi_{\ell-2,\ell-1} \\ & & & 0 \end{pmatrix} + \frac{z}{s(1-z)}\begin{pmatrix} \xi_{0,0} & & & O \\ \xi_{1,0} & \xi_{1,1} & & \\ \vdots & \ddots & \ddots & \\ \xi_{\ell-1,0} & \xi_{\ell-1,1} & \cdots & \xi_{\ell-1,\ell-1} \end{pmatrix}.$$

Finally, all the differential equations appearing in Sect. 7.1 emerge as the integrability condition of the system (7.29) and (7.30).

Remark 7.7. If $\ell = 2$, this system is equivalent to the Lax pair for P_V given in [15].

8 From UC hierarchy to Painlevé VI chain

In this section, we consider a certain similarity reduction of the (ℓ, ℓ)-periodic mUC hierarchy. The resulting system of nonlinear ordinary differential equations describes a monodromy preserving deformation of an $\ell \times \ell$ Fuchsian system with four regular singularities and, thus, it is regarded as a generalization of the sixth Painlevé equation.

8.1 Similarity reduction

Fix an integer $\ell \geq 2$. Let us require a solution $\tau_{m,n} = \tau_{m,n}(x, y)$ of the mUC hierarchy to fulfill the following conditions: (i) homogeneity $E\tau_{m,n} = d_{m,n}\tau_{m,n}$ ($d_{m,n} \in \mathbb{C}$); (ii) (ℓ, ℓ)-periodicity $\tau_{m+\ell,n} = \tau_{m,n+\ell} = \tau_{m,n}$; (iii) specialization of variables as

$$x_n = \frac{a + bt^n}{n} \quad \text{and} \quad y_n = \frac{a + bt^{-n}}{n}. \tag{8.1}$$

Concerning (i), we note that $d_{m,n} + d_{m+1,n+1} = d_{m,n+1} + d_{m+1,n}$ automatically holds. Under these conditions, set $\sigma_{m,n}(a, b, t) = \tau_{m,n}(x, y)$. First we prepare a system of bilinear relations satisfied by $\sigma_{m,n}$. To simplify an expression, we shall write for brevity $\overline{\sigma}_{m,n} = \sigma_{m,n}(a + 1, b, t)$ and $\underline{\sigma}_{m,n} = \sigma_{m,n}(a - 1, b, t)$, while $\sigma_{m,n} = \sigma_{m,n}(a, b, t)$. Likewise let $\check{\sigma}_{m,n}$ and $\underset{\smile}{\sigma}_{m,n}$ stand for $\sigma_{m,n}(a, b + 1, t)$ and $\sigma_{m,n}(a, b - 1, t)$, respectively.

Proposition 8.1. *The functions $\sigma_{m,n} = \sigma_{m,n}(a, b, t)$ satisfy the following bilinear equations*:

$$(t - 1)\sigma_{m,n}\check{\overline{\sigma}}_{m+1,n+1} - t\check{\sigma}_{m+1,n}\overline{\sigma}_{m,n+1} + \overline{\sigma}_{m+1,n}\check{\sigma}_{m,n+1} = 0, \tag{8.2}$$

$$(tD_t + b)\,\sigma_{m+1,n} \cdot \sigma_{m,n+1} = b\underset{\smile}{\sigma}_{m,n}\check{\sigma}_{m+1,n+1}, \tag{8.3}$$

$$(-tD_t + d_{m+1,n} - d_{m,n+1} + a)\,\sigma_{m+1,n} \cdot \sigma_{m,n+1} = a\underline{\sigma}_{m,n}\overline{\sigma}_{m+1,n+1}, \tag{8.4}$$

$$((t - 1)D_t - b)\,\underline{\sigma}_{m,n} \cdot \sigma_{m+1,n} + b\underset{\smile}{\sigma}_{m,n}\check{\underline{\sigma}}_{m+1,n} = 0, \tag{8.5}$$

$$(t(t - 1)D_t + (t - 1)(d_{m+1,n} - d_{m,n}) - a)\,\underset{\smile}{\sigma}_{m,n} \cdot \sigma_{m+1,n} + a\underline{\sigma}_{m,n}\overline{\sigma}_{m+1,n} = 0. \tag{8.6}$$

Proof. First (8.2) is immediate from (6.9) with $s = 1$. We see that (8.1) implies

$$t\frac{\mathrm{d}}{\mathrm{d}t} = b\delta_t \quad \text{and} \quad E = a\delta_1 + b\delta_t. \tag{8.7}$$

Hence, (8.3) and (8.5) are direct consequences of (6.10) and (6.11) with $s = 1$, respectively. By virtue of the homogeneity condition $E\tau_{m,n} = d_{m,n}\tau_{m,n}$, also (8.4) and (8.6) can be deduced from respectively (6.10) and (6.11) with (t, s) replaced by $(1, t)$. $\qquad\square$

Next we shall write down a system of nonlinear differential equations for appropriately chosen variables. Consider the functions $f_{m,n}^{(i)} = f_{m,n}^{(i)}(a, b, t)$ and $g_{m,n}^{(i)} = g_{m,n}^{(i)}(a, b, t)$ ($i = 0, 1$) defined by

$$f_{m,n}^{(0)} = \frac{\overline{\sigma}_{m,n-1}\sigma_{m-1,n-1}}{\overline{\sigma}_{m-1,n}\sigma_{m,n-2}}, \quad f_{m,n}^{(1)} = \frac{\check{\sigma}_{m,n-1}\sigma_{m-1,n-1}}{\check{\sigma}_{m-1,n}\sigma_{m,n-2}}, \tag{8.8}$$

$$g_{m,n}^{(0)} = a\frac{\sigma_{m-1,n-1}\overline{\sigma}_{m,n}}{\sigma_{m-1,n}\sigma_{m,n-1}}, \quad g_{m,n}^{(1)} = b\frac{\sigma_{m-1,n-1}\check{\sigma}_{m,n}}{\sigma_{m-1,n}\sigma_{m,n-1}}. \tag{8.9}$$

One can find the following conserved quantities

$$\prod_{j=1}^{\ell} f_{m+j,n-j}^{(i)} = 1, \tag{8.10}$$

$$\sum_{j=1}^{\ell} g_{m+j,n-j}^{(0)} = \ell a, \quad \sum_{j=1}^{\ell} g_{m+j,n-j}^{(1)} = \ell b. \tag{8.11}$$

Here the first line is immediate by definition of f-variables and the second can be verified from (8.3) and (8.4). Furthermore we introduce auxiliary variables $U_{m,n}^{(i,j)}$ and $V_{m,n}^{(i,j)}$ ($i, j \in \{0, 1\}$, $i \neq j$) given as

$$U_{m,n}^{(0,1)} = \frac{at}{1-t} \frac{\check{\sigma}_{m,n-1}\overline{\sigma}_{m,n}}{\sigma_{m,n-1}\check{\sigma}_{m,n}}, \quad U_{m,n}^{(1,0)} = \frac{b}{t-1} \frac{\overline{\underline{\sigma}}_{m,n-1}\check{\sigma}_{m,n}}{\sigma_{m,n-1}\overline{\sigma}_{m,n}}, \tag{8.12}$$

$$V_{m,n}^{(0,1)} = \frac{a}{1-t} \frac{\check{\sigma}_{m-1,n}\overline{\sigma}_{m,n}}{\sigma_{m-1,n}\check{\sigma}_{m,n}}, \quad V_{m,n}^{(1,0)} = \frac{bt}{t-1} \frac{\overline{\underline{\sigma}}_{m-1,n}\check{\sigma}_{m,n}}{\sigma_{m-1,n}\overline{\sigma}_{m,n}}. \tag{8.13}$$

Then we observe that

$$V_{m,n}^{(i,j)} - U_{m,n}^{(i,j)} = g_{m,n}^{(i)} \quad \text{and} \quad \frac{U_{m-1,n}^{(i,j)}}{V_{m,n-1}^{(i,j)}} = \frac{t_j f_{m,n}^{(j)}}{t_i f_{m,n}^{(i)}} \tag{8.14}$$

with $(t_0, t_1) = (1, t)$. Here the former is a consequence of (8.2) and the latter follows just from (8.8). Solving the above linear equations leads us to

$$U_{m,n}^{(i,j)} = \frac{1}{\left(\frac{t_i}{t_j}\right)^\ell - 1} \sum_{\alpha=1}^{\ell} g_{m-\alpha+1,n+\alpha-1}^{(i)} \prod_{\beta=1}^{\alpha-1} \frac{t_i f_{m-\beta+1,n+\beta}^{(i)}}{t_j f_{m-\beta+1,n+\beta}^{(j)}},$$

$$V_{m,n}^{(i,j)} = \frac{1}{\left(\frac{t_i}{t_j}\right)^\ell - 1} \sum_{\alpha=1}^{\ell} g_{m-\alpha,n+\alpha}^{(i)} \prod_{\beta=0}^{\alpha-1} \frac{t_i f_{m-\beta,n+\beta+1}^{(i)}}{t_j f_{m-\beta,n+\beta+1}^{(j)}}.$$

Namely, in view of (8.10), we can actually express $U_{m,n}^{(i,j)}$ and $V_{m,n}^{(i,j)}$ as polynomials in $f_{m,n}^{(i)}$ and $g_{m,n}^{(i)}$.

Theorem 8.2. *The functions $f_{m,n}^{(i)}$ and $g_{m,n}^{(i)}$ ($i = 0, 1$) satisfy the system of ordinary differential equations*

$$t\frac{df_{m,n}^{(1)}}{dt} = \left(\alpha_{m,n} - g_{m,n-1}^{(1)} + U_{m-1,n}^{(0,1)} - V_{m,n-1}^{(0,1)}\right) f_{m,n}^{(1)}, \tag{8.15a}$$

$$t\frac{df_{m,n}^{(0)}}{dt} = \left(-g_{m,n-1}^{(1)} - U_{m-1,n}^{(1,0)} + V_{m,n-1}^{(1,0)}\right) f_{m,n}^{(0)}, \tag{8.15b}$$

$$t\frac{dg_{m,n}^{(1)}}{dt} = -U_{m,n}^{(1,0)} g_{m,n}^{(0)} - V_{m,n}^{(0,1)} g_{m,n}^{(1)}, \tag{8.15c}$$

$$t\frac{dg_{m,n}^{(0)}}{dt} = U_{m,n}^{(0,1)} g_{m,n}^{(1)} + V_{m,n}^{(1,0)} g_{m,n}^{(0)}, \tag{8.15d}$$

where

$$\alpha_{m,n} = a + b + d_{m,n-1} - d_{m-1,n} = g_{m,n}^{(0)} + g_{m,n}^{(1)} \in \mathbb{C} \tag{8.16}$$

are constant parameters.

Proof. We shall demonstrate only (8.15a) and (8.15d) because the others can be verified in a similar manner. First we notice from (8.2) that the formulae

$$\check{g}_{m,n}^{(0)} = V_{m,n-1}^{(0,1)} - U_{m-1,n}^{(0,1)} \quad \text{and} \quad \overline{g}_{m,n}^{(1)} = V_{m,n-1}^{(1,0)} - U_{m-1,n}^{(1,0)} \tag{8.17}$$

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hold. Taking the logarithmic derivative of $f_{m,n}^{(1)}$ shows

$$
\begin{aligned}
\frac{t}{f_{m,n}^{(1)}} \frac{\mathrm{d} f_{m,n}^{(1)}}{\mathrm{d}t} &= \frac{t D_t \check{\sigma}_{m,n-1} \cdot \check{\sigma}_{m-1,n}}{\check{\sigma}_{m,n-1} \check{\sigma}_{m-1,n}} - \frac{t D_t \sigma_{m,n-2} \cdot \sigma_{m-1,n-1}}{\sigma_{m,n-2} \sigma_{m-1,n-1}} \\
&= \left(d_{m,n-1} - d_{m-1,n} + a - a \frac{\check{\sigma}_{m-1,n-1} \overline{\check{\sigma}}_{m,n}}{\check{\sigma}_{m,n-1} \check{\sigma}_{m-1,n}} \right) \\
&\quad + \left(b - b \frac{\underline{\sigma}_{m-1,n-2} \check{\sigma}_{m,n-1}}{\sigma_{m,n-2} \sigma_{m-1,n-1}} \right), \quad \text{using (8.3) and (8.4)} \\
&= \alpha_{m,n} - g_{m,n-1}^{(1)} - \check{g}_{m,n}^{(0)},
\end{aligned}
$$

which thus implies (8.15a). Likewise we observe that

$$
\begin{aligned}
\frac{t}{g_{m,n}^{(0)}} \frac{\mathrm{d} g_{m,n}^{(0)}}{\mathrm{d}t} &= \frac{t D_t \underline{\sigma}_{m-1,n-1} \cdot \sigma_{m,n-1}}{\underline{\sigma}_{m-1,n-1} \sigma_{m,n-1}} - \frac{t D_t \sigma_{m-1,n} \cdot \overline{\sigma}_{m,n}}{\sigma_{m-1,n} \overline{\sigma}_{m,n}} \\
&= \frac{bt}{t-1} \left(-\frac{\underline{\sigma}_{m-1,n-1} \check{\underline{\sigma}}_{m,n-1}}{\underline{\sigma}_{m-1,n-1} \sigma_{m,n-1}} + \frac{\overline{\sigma}_{m-1,n} \check{\sigma}_{m,n}}{\sigma_{m-1,n} \overline{\sigma}_{m,n}} \right), \quad \text{using (8.5)} \\
&= \frac{bt}{t-1} \left(-\frac{a g_{m,n}^{(1)}}{b g_{m,n}^{(0)}} \frac{\check{\underline{\sigma}}_{m,n-1} \overline{\sigma}_{m,n}}{\sigma_{m,n-1} \check{\sigma}_{m,n}} + \frac{\overline{\underline{\sigma}}_{m-1,n} \check{\sigma}_{m,n}}{\sigma_{m-1,n} \overline{\sigma}_{m,n}} \right) \\
&= U_{m,n}^{(0,1)} \frac{g_{m,n}^{(1)}}{g_{m,n}^{(0)}} + V_{m,n}^{(1,0)},
\end{aligned}
$$

which coincides with (8.15d). $\qquad \square$

Remark that for each (m, n) given the system is closed with respect to the 2ℓ-tuple of variables:

$$
g_{m+j,n-j} = g_{m+j,n-j}^{(1)} \quad \text{and} \quad h_{m+j,n-j+1} = \frac{f_{m+j,n-j+1}^{(1)}}{f_{m+j,n-j+1}^{(0)}} \quad \text{for } j \in \mathbb{Z}/\ell\mathbb{Z}.
$$

If we take into account the conservation (8.10) and (8.11), then the essential dimension of the phase space turns out to be $2\ell - 2$. As demonstrated below the case $\ell = 2$ is equivalent to the sixth Painlevé equation P_{VI}, which is, needless to say, of second order. We shall call (8.15) the P_{VI}-*chain*.

8.2 Example: $(2, 2)$-periodic case and P_{VI}

Consider the case where $\ell = 2$. We are interested in a system of differential equations satisfied by the variables $g = g_{1,0}^{(1)}$ and $h = f_{1,1}^{(1)}/f_{1,1}^{(0)}$. From Theorem 8.2 we observe that

$$
(t^2 - 1) \frac{\mathrm{d}g}{\mathrm{d}t} = -(h - h^{-1}) g(g - 2b) + (\alpha_{1,0} h + \alpha_{0,1} h^{-1}) g - 2\alpha_{1,0} b h,
$$

$$
(t^2 - 1) \frac{\mathrm{d}h}{\mathrm{d}t} = (h - t)(h - t^{-1})(2g - 2b - \alpha_{1,0}) - (\alpha_{0,0} t + \alpha_{1,1} t^{-1}) h + 2(a + b).
$$

Let us take the change of variables $(g, h) \mapsto (q, p)$ defined by

$$q = \frac{h(g - 2b)}{t(g + 2a - \alpha_{1,0})} = \frac{-b}{at} \frac{\check{\sigma}_{1,0} \underline{\sigma}_{1,0}}{\overline{\sigma}_{1,0} \underline{\sigma}_{1,0}},$$

$$p = \frac{g}{2q} = \frac{-at}{2} \frac{\overline{\sigma}_{1,0} \underline{\sigma}_{1,0} \check{\sigma}_{0,1}}{\sigma_{0,0} \sigma_{1,1} \check{\sigma}_{1,0}}.$$

Put $s = 1/t^2$. We then arrive at the Hamiltonian system

$$\frac{\mathrm{d}q}{\mathrm{d}s} = \frac{\partial H}{\partial p} \quad \text{and} \quad \frac{\mathrm{d}p}{\mathrm{d}s} = -\frac{\partial H}{\partial q}$$

with the Hamiltonian function $H = H(q, p; s)$ given as

$$s(s - 1)H = q(q - 1)(q - s)p^2 - ((\kappa_0 - 1)q(q - 1) + \kappa_1 q(q - s) + \theta(q - 1)(q - s))\, p + \kappa q,$$

where $\kappa_0 = d_{1,0} - d_{0,0} + 1/2$, $\kappa_1 = d_{0,1} - d_{0,0} + 1/2$, $\theta = b$, and $\kappa = \alpha_{1,0}(\alpha_{1,0} - 2a)/4$. This is exactly the Hamiltonian form of P_{VI}; see [24, 35].

Note that also the P_{VI}-chain, (8.15), for a general ℓ case can be transformed into a Hamiltonian system whose Hamiltonian function is a polynomial in the canonical variables. For details to [51].

8.3 Lax formalism

The homogeneity $E\tau_{m,n} = d_{m,n}\tau_{m,n}$ ensures the formula $(E - k\partial/\partial k)\psi_{m,n} = (d_{m,n-1} - d_{m,n})\psi_{m,n}$; see Lemma 7.5. Set $\phi_{m,n}(a, b, t, k) = \psi_{m,n}(\boldsymbol{x}, \boldsymbol{y}, k)$ under the substitution (8.1). Then, Lemma 6.8 together with (8.7) leads us to the

Lemma 8.3. *The wave functions* $\phi_{m,n} = \phi_{m,n}(a, b, t, k)$ *satisfy the following linear equations*:

$$\phi_{m,n} = \frac{1}{f_{m+1,n+1}^{(0)}} \overline{\phi}_{m,n+1} - k\overline{\phi}_{m+1,n}, \tag{8.18}$$

$$\phi_{m,n} = \frac{1}{f_{m+1,n+1}^{(1)}} \check{\phi}_{m,n+1} - tk\check{\phi}_{m+1,n}, \tag{8.19}$$

$$t\frac{\partial}{\partial t}\phi_{m,n} = \left(g_{m+1,n}^{(1)} - b\right)\phi_{m,n} + tkg_{m+1,n}^{(1)}\check{\phi}_{m+1,n}, \tag{8.20}$$

$$\left(k\frac{\partial}{\partial k} - t\frac{\partial}{\partial t}\right)\phi_{m,n} = (d_{m,n} - d_{m,n-1})\phi_{m,n} + \left(g_{m+1,n}^{(0)} - a\right)\phi_{m,n} + kg_{m+1,n}^{(0)}\overline{\phi}_{m+1,n}. \tag{8.21}$$

Here we have used the abbreviations $\overline{\phi}_{m,n} = \phi_{m,n}(a + 1, b, t, k)$ *and* $\check{\phi}_{m,n} = \phi_{m,n}(a, b + 1, t, k)$.

Due to the (ℓ, ℓ)-periodicity one can solve the linear equations (8.18) for $\overline{\phi}_{m,n}$; thus,

$$\overline{\phi}_{m,n} = \frac{1}{1 - k^\ell} \sum_{j=1}^{\ell} k^{j-1} \left(\prod_{i=1}^{j} f_{m+i,n-i+1}^{(0)}\right) \phi_{m+j-1,n-j}.$$

Likewise (8.19) tells us that

$$\check{\phi}_{m,n} = \frac{1}{1 - (tk)^\ell} \sum_{j=1}^{\ell} (tk)^{j-1} \left(\prod_{i=1}^{j} f_{m+i,n-i+1}^{(1)}\right) \phi_{m+j-1,n-j}.$$

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Firstly, we put our attention to the linear differential equation with respect to the spectral variable k. Noticing (8.16) and combining (8.20) with (8.21), we therefore obtain

$$k\frac{\partial}{\partial k}\phi_{m,n} = (d_{m+1,n-1} - d_{m,n-1})\phi_{m,n} + \frac{g^{(0)}_{m+1,n}}{1 - k^{\ell}}\sum_{j=1}^{\ell}k^j\left(\prod_{i=1}^{j}f^{(0)}_{m+i+1,n-i+1}\right)\phi_{m+j,n-j}$$

$$+ \frac{g^{(1)}_{m+1,n}}{1 - (tk)^{\ell}}\sum_{j=1}^{\ell}(tk)^j\left(\prod_{i=1}^{j}f^{(1)}_{m+i+1,n-i+1}\right)\phi_{m+j,n-j},$$

which has the $2\ell + 2$ regular singularities at $k = 0, \infty, \exp\left(2\pi\sqrt{-1}n/\ell\right), t^{-1}\exp\left(2\pi\sqrt{-1}n/\ell\right)$ ($n \in \mathbb{Z}/\ell\mathbb{Z}$). But, however, this expression is quite redundant. Let

$$\Phi = {}^{\mathrm{T}}\left(\phi_{0,0}, k\phi_{1,-1}, k^2\phi_{2,-2}, \ldots, k^{\ell-1}\phi_{\ell-1,-\ell+1}\right), \quad z = k^{\ell}, \quad \text{and} \quad s = 1/t^{\ell}.$$

Then we obtain the equation

$$\frac{\partial\Phi}{\partial z} = A\Phi = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_s}{z-s}\right)\Phi, \tag{8.22}$$

where the $\ell \times \ell$ matrices A_0, A_1, and A_s read

$$A_0 = \begin{pmatrix} e_0 & w_{0,1} & \cdots & w_{0,\ell-1} \\ & e_1 & \ddots & \vdots \\ & & \ddots & w_{\ell-2,\ell-1} \\ & & & e_{\ell-1} \end{pmatrix},$$

$$A_1 = -\left(v^{(0)}_{i,j}\right)_{0 \le i,j \le \ell-1},$$

$$A_s = -\begin{pmatrix} 0 & v^{(1)}_{0,1} & \cdots & v^{(1)}_{0,\ell-1} \\ & 0 & \ddots & \vdots \\ & & \ddots & v^{(1)}_{\ell-2,\ell-1} \\ & & & 0 \end{pmatrix} - s\begin{pmatrix} v^{(1)}_{0,0} & & & O \\ v^{(1)}_{1,0} & v^{(1)}_{1,1} & & \\ \vdots & \ddots & \ddots & \\ v^{(1)}_{\ell-1,0} & v^{(1)}_{\ell-1,1} & \cdots & v^{(1)}_{\ell-1,\ell-1} \end{pmatrix}$$

with $e_n = (d_{n+1,-n-1} - d_{n,-n-1} + n)/\ell$ and $w_{i,j} = v^{(0)}_{i,j} + v^{(1)}_{i,j}$; we set

$$v^{(0)}_{n,n+j} = \frac{g^{(0)}_{n+1,-n}}{\ell}\prod_{i=1}^{j}f^{(0)}_{n+i+1,-n-i+1},$$

$$v^{(1)}_{n,n+j} = \frac{g^{(1)}_{n+1,-n}}{\ell}t^j\prod_{i=1}^{j}f^{(1)}_{n+i+1,-n-i+1}$$

for $0 \le n \le \ell - 1$ and $1 \le j \le \ell$. Note that we read appropriately the suffixes of dependent variables modulo ℓ as before. The equation (8.22) still remains Fuchsian and has the four regular singularities $z = 0, 1, s, \infty$. The exponents at each singularity are listed in the following table

(Riemann scheme):

$$
\begin{array}{c|c}
\text{Singularity} & \text{Exponents} \\
\hline
z = 0 & (e_0, e_1, \ldots, e_{\ell-1}) \\
z = 1 & (-a, 0, \ldots, 0) \\
z = s & (-b, 0, \ldots, 0) \\
z = \infty & \left(\frac{\alpha_{1,0}}{\ell} - e_0, \frac{\alpha_{2,-1}}{\ell} - e_1, \ldots, \frac{\alpha_{\ell,-\ell+1}}{\ell} - e_{\ell-1} \right)
\end{array}
\tag{8.23}
$$

Note that both A_1 and A_s are not full rank (but rank one), unlike A_0 and $A_\infty = -(A_0 + A_1 + A_s)$. For example let us look at A_s. Fix the row vectors

$$
\boldsymbol{f} = \left(t^j \prod_{i=1}^{j} f^{(1)}_{i+1,-i+1} \right)_{0 \le j \le \ell-1} = \left(1, t f^{(1)}_{2,0}, t^2 f^{(1)}_{2,0} f^{(1)}_{3,-1}, \ldots, t^{\ell-1} f^{(1)}_{2,0} f^{(1)}_{3,-1} \cdots f^{(1)}_{0,2} \right),
$$

$$
\boldsymbol{g} = \left(\frac{g^{(1)}_{j+1,-j}}{t^j \prod_{i=1}^{j} f^{(1)}_{i+1,-i+1}} \right)_{0 \le j \le \ell-1} = \left(g^{(1)}_{1,0}, \frac{g^{(1)}_{2,-1}}{t f^{(1)}_{2,0}}, \frac{g^{(1)}_{3,-2}}{t^2 f^{(1)}_{2,0} f^{(1)}_{3,-1}}, \ldots, \frac{g^{(1)}_{\ell,-\ell+1}}{t^{\ell-1} f^{(1)}_{2,0} f^{(1)}_{3,-1} \cdots f^{(1)}_{0,2}} \right).
$$

We see indeed that A_s is expressed as $-\ell A_s = {}^{\mathrm{T}}\boldsymbol{g} \cdot \boldsymbol{f}$ and that $\boldsymbol{f} \cdot {}^{\mathrm{T}}\boldsymbol{g} = \sum_{j=0}^{\ell-1} g^{(1)}_{j+1,-j} = \ell b$. Also, the case of A_1 can be checked in the same way.

Secondly, we obtain from (8.20) the deformation equation with respect to $s = 1/t^\ell$:

$$
\frac{\partial}{\partial s} \Phi = B\Phi,
\tag{8.24}
$$

where

$$
B = \operatorname{diag}\left(\frac{b}{\ell s} - v^{(1)}_{i,i} \right)_{0 \le i \le \ell-1}
$$
$$
+ \frac{1}{z-s} \begin{pmatrix} 0 & v^{(1)}_{0,1} & \cdots & v^{(1)}_{0,\ell-1} \\ & 0 & \ddots & \vdots \\ & & \ddots & v^{(1)}_{\ell-2,\ell-1} \\ & & & 0 \end{pmatrix} + \frac{z}{z-s} \begin{pmatrix} v^{(1)}_{0,0} & & & O \\ v^{(1)}_{1,0} & v^{(1)}_{1,1} & & \\ \vdots & \ddots & \ddots & \\ v^{(1)}_{\ell-1,0} & v^{(1)}_{\ell-1,1} & \cdots & v^{(1)}_{\ell-1,\ell-1} \end{pmatrix}.
$$

Finally, we can again recover the P_{VI}-chain from the integrability condition $\left[\frac{\partial}{\partial s} - B, \frac{\partial}{\partial z} - A \right] = 0$ of the system (8.22) and (8.24). Since (8.22) is Fuchsian, the P_{VI}-chain turns out to be equivalent to a Schlesinger system ([43]) specified by the Riemann scheme (8.23).

Remark 8.4. If $\ell = 2$, the above system is equivalent to the Lax pair for P_{VI} given in [15, 43].

A From KP hierarchy to Painlevé I equation

In this appendix, we demonstrate the derivation of the first Painlevé equation (P_{I}) from the two-reduced KP hierarchy, i.e., KdV hierarchy, through a certain reduction procedure by the use of a Virasoro operator. Such a relationship between P_{I} and the KP hierarchy was first recognized in the context of 'two-dimensional quantum gravity'; see, e.g., [2, 12, 22]. We also show that the Lax pair for P_{I} naturally arises from the associated linear equations of the KP hierarchy.

A.1 Reduction by using a Virasoro operator

Introduce a differential operator (a *Virasoro operator*):

$$L(= L_{-2}) = \frac{x_1{}^2}{2} + \sum_{n=1}^{\infty} (n+2)x_{n+2}\frac{\partial}{\partial x_n}.$$

Let $\tau = \tau(x)$ be a solution of the KP hierarchy, (2.5) or (2.6), that fulfills the following conditions:

$$\frac{\partial \tau}{\partial x_{2n}} \equiv 0 \quad (n = 1, 2, \ldots), \tag{A.1}$$

$$L\tau(x) = c\tau(x) \quad (c \in \mathbb{C}). \tag{A.2}$$

The first condition means that τ is a solution of the KdV hierarchy. For instance, it satisfies the KdV equation (cf. (2.7)):

$$\left(D_{x_1}{}^4 - 4D_{x_1}D_{x_3}\right)\tau \cdot \tau = 0. \tag{A.3}$$

Set $\sigma(x) = \tau(x)$ under the specialization

$$x_1 = x, \quad x_5 = \frac{4}{5}, \quad \text{and} \quad x_n = 0 \quad (n \neq 1, 5). \tag{A.4}$$

Proposition A.1. *A function* $\sigma = \sigma(x)$ *satisfies the bilinear differential equation*

$$\left(D_x{}^4 + 2x\right)\sigma \cdot \sigma = 0. \tag{A.5}$$

Proof. Observing that

$$\frac{\partial}{\partial x_1} = \frac{d}{dx} \quad \text{and} \quad 4\frac{\partial}{\partial x_3} = -\frac{x^2}{2} + L, \tag{A.6}$$

we can deduce (A.5) from (A.3) immediately. □

If we take the variable $q = -(d/dx)^2 \log \sigma$, then (A.5) is equivalently rewritten into the first Painlevé equation P_{I}:

$$\frac{d^2 q}{dx^2} = 6q^2 + x.$$

A.2 Lax formalism

We first prepare some formulae for the wave function. Set

$$\rho(x, k) = X^+(k)\tau(x) = \tau(x - [k^{-1}])e^{\xi(x,k)}.$$

The KP hierarchy (2.5) multiplied by $1 \otimes X^+(k)$ yields $\sum_{i+j=-2} X_i^- \tau \otimes X_j^+ \rho = 0$, namely,

$$G_1(\tau(x), \rho(x, k); u) = 0;$$

recall (2.25). The coefficients of $1 = u^0$ and u_1 of the above equation show respectively that

$$\left(D_{x_1}{}^2 + D_{x_2}\right)\tau \cdot \rho = 0,$$

$$\left(D_{x_1}{}^3 - 3D_{x_1}D_{x_2} - 4D_{x_3}\right)\tau \cdot \rho = 0.$$

Hence, for the wave function $\psi(\boldsymbol{x}, k) = \rho(\boldsymbol{x}, k)/\tau(\boldsymbol{x})$, we find that

$$\left(\frac{\partial^2}{\partial x_1{}^2} - \frac{\partial}{\partial x_2} + \frac{D_{x_1}{}^2 \tau \cdot \tau}{\tau^2}\right)\psi = 0, \tag{A.7}$$

$$\left(\frac{\partial^3}{\partial x_1{}^3} + 3\frac{\partial^2}{\partial x_1 \partial x_2} - 4\frac{\partial}{\partial x_3} + 3\frac{D_{x_1}{}^2 \tau \cdot \tau}{\tau^2}\frac{\partial}{\partial x_1} + 3\frac{D_{x_1} D_{x_2} \tau \cdot \tau}{\tau^2}\right)\psi = 0. \tag{A.8}$$

Now, by applying the constraint (A.2), we can verify the formula

$$\left(L - \frac{x_1{}^2}{2} - k^{-1}\frac{\partial}{\partial k}\right)\psi = \left(-2x_2 - \frac{k^{-2}}{2}\right)\psi. \tag{A.9}$$

Under the substitution (A.4), we write $\phi(x, k) = \psi(\boldsymbol{x}, k)$. By virtue of (A.9) together with (A.6), the lemma below follows readily from (A.7) and (A.8).

Lemma A.2. *The wave function $\phi = \phi(x, k)$ satisfies the following linear equations*:

$$k^{-1}\frac{\partial \phi}{\partial k} = \left(2p + \frac{k^{-2}}{2}\right)\phi + 4\left(-q + k^2\right)\frac{\partial \phi}{\partial x},$$

$$\frac{\partial^2 \phi}{\partial x^2} = \left(2q + k^2\right)\phi,$$

where $p = \mathrm{d}q/\mathrm{d}x$.

Let $\Phi = {}^{\mathrm{T}}(\phi, \partial\phi/\partial x)\, k^{-1/2}$ and $z = k^2$. We then arrive at the system of 2×2 matrix equations

$$\frac{\partial}{\partial z}\Phi = A\Phi = \left(\begin{pmatrix} p & -2q \\ 2q^2 + x & -p \end{pmatrix} + z\begin{pmatrix} 0 & 2 \\ 2q & 0 \end{pmatrix} + z^2\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\right)\Phi, \tag{A.10}$$

$$\frac{\partial}{\partial x}\Phi = B\Phi = \left(\begin{pmatrix} 0 & 1 \\ 2q & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)\Phi. \tag{A.11}$$

We see that (A.10) has only one irregular singularity at $z = \infty$ whose Poincaré rank is $5/2$. The latter, (A.11), governs the monodromy preserving deformation of the former. This system is identical to the Lax pair for P_{I} given in [15]. Indeed, P_{I} can be recovered as its integrability condition.

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